A finite element formulation for finite strain elasto-plastic analysis based on mixed interpolation of tensorial components

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Abstract

A quadrilateral finite element formulation for modelling finite strain elasto-plastic deformation processes is presented. The developed formulation is based on Lee’s multiplicative decomposition of the deformation gradient and on the hyperelastic expression of the associated Von Mises flow rule. The Hencky or logarithmic strain measure is used in the strain energy function. The quadrilateral element is developed using displacements interpolation and interpolation of the Hencky strain tensor covariant components. A consistent linearization is developed, obtaining a symmetric stiffness matrix.

1. Introduction

Accurate modelling of finite strain elasto-plastic deformation processes has been an important goal for researchers in the computational mechanics field for quite a long time. Only in the 1980s has the state of the art for this type of analysis been firmly established, mainly due to the work of Lee [1, 2], Argyris and Doltsinis [3, 4] and Simo and Ortiz [5–7].

Nowadays most researchers agree on the fact that accurate and efficient modelling of finite strain elasto-plastic deformation of metals is based on

- Lee’s multiplicative decomposition of the deformation gradient;
- the hyperelastic expression of the Von Mises flow theory developed using the principle of maximum plastic dissipation [3–7] (associated flow rule [8]).

The use of the more traditional hypoelastic formulation is compatible with the above scheme if the additive decomposition of the strain rate tensor follows from the multiplicative decomposition of the deformation gradient and if the appropriate constitutive equations are used [9].

In Section 2, we develop a total Lagrangian–Hencky formulation (TLH) using a hyperelastic constitutive equation in terms of Hencky’s logarithmic strain tensor and its work conjugate stress tensor [10, 11]. The reason for this choice is that according to the experimental data reported by Anand [12]: ‘the classical strain energy function of infinitesimal isotropic elasticity is in good agreement with experiment for a wide class of materials for moderately large deformations, provided the infinitesimal strain measure occurring in the strain energy function is replaced by the Hencky or logarithmic measure of finite strain’.

From a numerical viewpoint, an additional advantage for using the Hencky strain measure is that the first invariant of the logarithmic strain tensor is the logarithmic volume strain; therefore many

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techniques developed for handling incompressibility in the infinitesimal strain problem can be carried over to the finite strain problem.

Other researchers who used the Hencky strain tensor for their finite strain elasto-plastic or viscoplastic formulations are: Rolph and Bathe [13]; Weber and Anand [14]; Eterovic and Bathe [15]; Peric, Owen and Honnor [16] and Simo [17].

In Section 3 we present the quadrilateral element developed using the TLH formulation and the technique of mixed interpolation of tensorial components [18–24]: the QMITC-TLH element.

In the QMITC-TLH element formulation we use

- a displacement interpolation;
- an interpolation for the covariant components of the total Hencky strain tensor referred to the
  natural coordinate system at the element center.

Both interpolations are tied together at appropriate sampling points.

It was shown that the equivalent formulation for infinitesimal strain problems [21, 22]:
- does not present spurious zero energy modes [25, 26];
- satisfies Irons’ patch test [25, 26];
- does not lock either in plane strain or axisymmetric situations [27];
- has high predictive capabilities and is as insensitive to element distortions as possible without
  violating the patch test [28] and without increasing the number of internal displacement modes to
  more than one [29].

In Section 4 we present an ad-hoc version of the well-known radial return algorithm [30] that is based on previous developments by Balmer et al. [31], Argyris and Doltzis [32], Ortiz and Popov [33], Simo and Taylor [34, 35], Ortiz and Simo [36] and Eterovic and Bathe [15].

In very recent works, Simo [17] and Simo and Meschke [37] presented a new version of the radial return algorithm for finite strain elasto-plastic problems. In this new version, the same piece of code used for the radial return algorithm for infinitesimal strain problems can be directly used for finite strain problems.

In Section 5, a symmetric algorithmic-consistent stiffness matrix [34] is derived and in Section 6 some numerical examples are analyzed in order to investigate the performance of the QMITC-TLH element.

2. The total Lagrangian–Hencky formulation

2.1. Basic definitions

For a continuum body $\mathcal{B}$ undergoing a deformation process like the one represented in Fig. 1, we can calculate the deformation gradient $\dot{\mathbf{F}}$, corresponding to time (load level) $t$ and referred to the reference configuration ($t = 0$). We can also perform the polar decomposition [38, 39] of the deformation gradient and obtain

$$
\dot{\mathbf{F}} = \dot{\mathbf{R}} \cdot \dot{\mathbf{U}} = \dot{\mathbf{V}} \cdot \dot{\mathbf{R}},
$$

(1)

where $\dot{\mathbf{R}}$ is the rotation tensor, $\dot{\mathbf{U}}$ is the right stretch tensor and $\dot{\mathbf{V}}$ is the left stretch tensor. We can also write [10]

$$
\dot{\mathbf{U}} = \dot{\mathbf{R}}_L \cdot \dot{\mathbf{A}} \cdot \dot{\mathbf{R}}_L^T,
$$

(2a)

$$
\dot{\mathbf{V}} = \dot{\mathbf{R}}_E \cdot \dot{\mathbf{A}} \cdot \dot{\mathbf{R}}_E^T.
$$

(2b)

In the above $\dot{\mathbf{A}}$ is the diagonal tensor whose components are the eigenvalues of $\dot{\mathbf{U}}$ ($\dot{\mathbf{V}}$); $\dot{\mathbf{R}}_L$ is a tensor that rotates from the reference coordinate system to the orthonormal system formed by the eigenvectors of $\dot{\mathbf{U}}$ and $\dot{\mathbf{R}}_E$ is a tensor that rotates from the reference coordinate system to the orthonormal system formed by the eigenvectors of $\dot{\mathbf{V}}$. From the above,

$$
\dot{\mathbf{R}}_E = \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}_L.
$$

(3)

The Hencky strain tensor is defined as
\[ \dot{\sigma} = \ln(\dot{\mathbf{U}}) = \dot{\mathbf{R}} \cdot \ln(\dot{\mathbf{A}}) \cdot \dot{\mathbf{R}}^T, \]  

(4)

and is co-linear with \( \dot{\mathbf{U}} \) and \( \dot{\mathbf{C}} = \dot{\mathbf{U}}^T \cdot \dot{\mathbf{U}}. \)

With \( \dot{\sigma} \) the Cauchy stress tensor in the spatial configuration, we define the Kirchhoff stress tensor in the same configuration as

\[ \dot{\tau} = \frac{\rho}{\dot{\rho}} \dot{\sigma}, \]  

(5)

where \( \rho \) is the density in the reference configuration and \( \dot{\rho} \) is the density in the spatial configuration.

Using the standard pull-back/push-forward notation of manifold analysis (e.g. [40]), we can define the pull-back of the Kirchhoff stress tensor under \( \dot{\mathbf{R}} \) [41]:

\[ \dot{\Gamma} = \dot{\mathbf{R}}^*\dot{\tau}. \]  

(6)

It was shown by Atluri [11] that for isotropic materials the stress-work rate per unit volume of the reference configuration is

\[ \dot{\mathbf{W}} = \dot{\Gamma} : \frac{d}{dt}(\dot{\mathbf{H}}), \]  

(7)

In an isotropic material, \( \dot{\sigma} \) is co-axial with \( \dot{\mathbf{V}} \) and therefore \( \dot{\Gamma} \) and \( \dot{\mathbf{H}} \) are also co-axial.

2.2. Kinematics of elasto-plastic deformations

For a solid continuum body \( \mathcal{B} \) undergoing an elasto-plastic deformation process, we present a scheme of Lee's multiplicative decomposition of the deformation gradient in Fig. 2 [1, 2]. The intermediate (unstressed) configuration does not need to be an actual configuration of the body \( \mathcal{B} \) because it does not need to be a smooth homeomorphism of \( \mathcal{B} \) onto a three-dimensional Euclidean space [38] and in general it will not be an actual configuration [1, 2].

For the multiplicative decomposition,

\[ \dot{\mathbf{F}} = \dot{\mathbf{F}}_e \cdot \dot{\mathbf{F}}_p. \]  

(8)

The velocity gradient is defined in the spatial configuration [38], and the following relation holds:
\[ t = 0 \]

**Fig. 2.** Lee’s multiplicative decomposition.

\[ 'I = \nabla_{i_x} ' \bar{x} = 'F \cdot 'F^{-1}. \] (9)

In the above the symbol \( \nabla_{i_x} \) means spatial gradient.

Using Eq. (8), we obtain

\[ 'I = 'F_e \cdot 'F_e^{-1} + 'F_p \cdot 'F_p^{-1}. \] (10)

We call \( 'I_p = 'F_e \cdot 'F_e^{-1} \), and we can re-write Eq. (10) as

\[ 'I = 'F_e \cdot 'F_e^{-1} + 'F_p \cdot 'F_p \cdot 'I_p \] (11)

Also we can make an additive decomposition of \( 'I_p \) into a symmetric tensor \( ('d_p) \) and a skew-symmetric one \( ('\omega_p) [1, 2] \).

If the material under consideration has isotropic elastic properties we can impose \( ('\omega_p = 0 \) [14, 15].

### 2.3. Stresses in elasto-plastic problems

By doing the polar decomposition of \( 'F_e \) and \( 'F_p \) we obtain

\[ 'F_e = 'R_e \cdot 'U_e = 'V_e \cdot 'R_e, \] (12a)

\[ 'F_p = 'R_p \cdot 'U_p = 'V_p \cdot 'R_p. \] (12b)

Therefore we define the elastic Hencky strain tensor as

\[ 'H_e = \ln ('U_e). \] (13)

We can now define

\[ '\Gamma = 'R_e \cdot ('\tau). \] (14)

### 2.4. The yield criteria

Following the work by Lee [1, 2] we formulate the yield criteria in terms of Kirchhoff stresses. Since we are interested in modelling elasto-plastic deformation processes in metals we use the Von Mises \( (J_2) \) yield criteria with combined isotropic/kinematic hardening [15]:

\[ '\phi = \left( \frac{3}{2} ('\tau_D - '\alpha) : ('\tau_D - '\alpha) \right)^{1/2} - '\sigma_p = 0, \] (15a)
where \( \tau_D \) is the deviatoric Kirchhoff stress tensor, \( \alpha \) is the back-stress tensor (traceless) and \( \sigma \) is the yield stress in the configuration at time \( t \).

It is important to remark that the tensors \( B = R^*(\alpha) \) and \( \Gamma_D = R^*(\tau_D) \) are traceless: using them we can also formulate the yield condition as

\[
\phi = \left( \frac{3}{2} (\Gamma_D - B) : (\Gamma_D - B) \right)^{1/2} - \sigma = 0 .
\] (15b)

We consider the following evolution equations (hardening) [15]:

\[
\dot{\sigma} = \beta h \dot{\varepsilon}_p, \quad \dot{B} = \frac{3}{2} (1 - \beta) h \dot{d}_p .
\] (16a,b)

where \( h(h(\dot{\varepsilon}_p)) \) is the hardening modulus and \( \beta \in [0,1] \) is the hardening ratio: \( \beta = 0 \) corresponds to purely kinematic hardening and \( \beta = 1 \) corresponds to purely isotropic hardening. In the above \( \dot{\varepsilon}_p \) is the equivalent plastic strain, to be defined in what follows.

2.5. Energy dissipation

We now introduce \( \psi \), the free energy at the spatial configuration per unit volume of the reference configuration.

Considering that the mechanical problem is uncoupled from the thermal problem, we can write the Clausius–Duhem inequality or principle of dissipation [36] as

\[
\dot{\tau} : \dot{d} - \dot{\psi} \geq 0 .
\] (17)

In the above,

\[
\dot{\psi} = \psi(\dot{\sigma}, \dot{\varepsilon}) .
\] (18)

Following Simo [6], we use the following uncoupled expression for the free energy:

\[
\psi = \psi_e(\dot{\sigma}, \dot{\varepsilon}) .
\] (19)

From Eq. (11)

\[
I = I_e + I_p ,
\] (20a)

\[
I_e = \dot{F} e \cdot \dot{F} e^{-1} = \dot{d}_e + \dot{\omega}_e ,
\] (20b)

\[
I_p = \dot{F} e \cdot \dot{I} e \cdot \dot{F} e^{-1} = \dot{d}_p + \dot{\omega}_p .
\] (20c)

Therefore,

\[
\dot{\tau} : \dot{d} = \dot{\tau} : (\dot{d}_e + \dot{d}_p) ,
\] (21a)

and after some algebra,

\[
\dot{\tau} : \dot{d}_p = (\dot{\tau} \cdot \dot{d}_p) : \dot{I}_p .
\] (21b)

Taking into account that for an elastically isotropic material [11], \( \dot{\tau} : \dot{d}_e = \dot{\Gamma} : \dot{F} e \dot{H}_e / dt \) we can write the Clausius–Duhem inequality as

\[
\left( \dot{\Gamma} - \frac{\partial \psi_e}{\partial \dot{\sigma}_e} \right) : \dot{d} (\dot{\sigma}_e) + (\dot{\tau} \cdot \dot{d}_p) : \dot{I}_p - \dot{\psi}_p \geq 0 .
\] (22)

Since the above must also be valid for the case of zero increment of plastic deformation, we obtain

\[
\dot{\Gamma} = \frac{\partial \psi_e}{\partial \dot{\sigma}_e} .
\] (23)

We define as dissipation [6],
\begin{equation}
\mathbf{D} = (\mathbf{F}_c^t \cdot \mathbf{r} \cdot \mathbf{F}_c^{-1})^{-1} \mathbf{I}_p - \mathbf{\dot{\psi}}_p \geq 0.
\end{equation}

Using Eq. (12a), considering that \( \dot{U}_e \) and \( \mathbf{I} \) are colinear for elastically isotropic materials and taking into account that the contraction of a symmetric tensor with a skew-symmetric tensor is zero, we obtain
\begin{equation}
\mathbf{D} = \mathbf{I} \cdot \mathbf{\dot{\psi}}_p = \mathbf{0}.
\end{equation}

For fixed plastic variables, we search for the value of \( \mathbf{I} \) that maximizes the dissipation, under the constraint \( \dot{\phi} \leq 0 \) \([6]\), obtaining the well known associated flow rule \([8]\). To solve the minimization problem we use the Kuhn–Tucker conditions \([42]\) and we obtain for plastic loading,
\begin{equation}
\mathbf{\dot{\psi}}_p = \frac{3}{2} \mathbf{\mu} \frac{(\mathbf{I}_D - \mathbf{I})}{\sqrt{\frac{3}{2} (\mathbf{I}_D - \mathbf{I}) : (\mathbf{I}_D - \mathbf{I})}},
\end{equation}

\begin{equation}
\mathbf{\mu} = \sqrt{\frac{2}{3}} \mathbf{\dot{\psi}}_p \cdot \mathbf{\dot{\psi}}_p = \mathbf{\dot{\varepsilon}}_p.
\end{equation}

where \( \mathbf{\dot{\varepsilon}}_p \) is the equivalent plastic strain rate.

We define a 'normal tensor'
\begin{equation}
\mathbf{N} = \frac{(\mathbf{I}_D - \mathbf{I})}{\sqrt{(\mathbf{I}_D - \mathbf{I}) : (\mathbf{I}_D - \mathbf{I})}},
\end{equation}

and calling \( \lambda = \sqrt{3/2} \mathbf{\mu} \) we obtain from Eqs. (26) the normality rule \([8]\),
\begin{equation}
\mathbf{\dot{\psi}}_p = \lambda \mathbf{N}.
\end{equation}

3. The QMITC-TLH formulation

Following \([21–23]\), we use for the QMITC-TLH formulation:
- displacements interpolation corresponding to the five nodes isoparametric element shown in Fig. 3(a);
- the degrees of freedom corresponding to the central node are going to be condensed at the element level;
- Total Hencky strain covariant components interpolation:
\begin{equation}
\mathbf{\dot{\varepsilon}}_n = \mathbf{\dot{\varepsilon}}_n^D + \frac{\sqrt{3}}{2} \sum_{i=1}^{3} \frac{J_0}{2} \left[ \mathbf{\dot{\varepsilon}}_n^{D,1} + \mathbf{\dot{\varepsilon}}_n^{D,2} + \mathbf{\dot{\varepsilon}}_n^{D,3} \right] + \sum_{i=1}^{3} \frac{\sqrt{3}}{2} \left[ \mathbf{\dot{\varepsilon}}_n^{L,1} + \mathbf{\dot{\varepsilon}}_n^{L,2} + \mathbf{\dot{\varepsilon}}_n^{L,3} \right],
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig_3.png}
\caption{QMITC-TLH element.}
\end{figure}
with \( i = r, s \) for plane stress and plane stress analysis and \( i = r, s, t \) for axisymmetric analysis.

\[
\left. \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0} = \left. \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} \tag{29b}
\]

In the above equations, \( \left. \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} \) is the covariant Hencky strain component calculated from the Displacements Interpolation at sampling point \( P \) and referred to the contravariant base vectors of the element natural coordinate system at \( O \) at \( t = 0 \) ("\( \hat{g}_{i} \)"). The sampling points A, B, C, D and O are indicated in Fig. 3(b). \( J \) is the element Jacobian at point \((r, s)\) for \( t = 0 \) and \( J_{0} \) is the element Jacobian at point \((0, 0)\) for \( t = 0 \).

### 3.1. Incompressibility constraint

In order to investigate the capability of the QMITC-TLH element to represent incompressible modes, we follow the ideas used for the infinitesimal strain case by Nagtegaal et al. [27].

For an incremental incompressible mode

\[
\text{tr}(\gamma_{i} \Delta \hat{H}) = \text{tr}(\gamma_{i} \hat{H}) \tag{30}
\]

Following what has been done previously for the QMITC element in the infinitesimal strain case [22], we analyze now the cases of plane strain and axisymmetric analyses.

#### 3.1.1. Plane strain analysis

For plane strain analysis

\[
\text{tr}(\gamma_{i} \hat{H}) = \left[ \gamma_{ij} \left( \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} \right] + \left[ \gamma_{ii}^{(2)} \left( \frac{\partial}{\partial \epsilon} \hat{H}_{ii} \right|_{0}^{Dl} - \left. \frac{\partial}{\partial \epsilon} \hat{H}_{ii} \right|_{0}^{Dl} \right] J_{U} \gamma + \left[ \gamma_{ij}^{(2)} \left( \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} - \left. \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} \right] J_{U} \gamma \tag{31}
\]

where

\[
J_{U} = \frac{\sqrt{3}}{2} \frac{J_{0}}{J} \
\text{and} \ i, j = 1, 2.
\]

Hence the fulfillment of Eq. (30) implies in the most general case three constraints.

In [27] it was demonstrated that when a mesh is uniformly refined

\[
\lim_{k \to \infty} \left( \frac{k}{p} \right) = \frac{n}{2} \tag{32a}
\]

where \( p \) is the number of elements, \( k \) is the number of interior nodes in the mesh and \( n \pi \) is the sum of the nodal angles of the element, in radians.

Therefore for the QMITC-TLH element, considering that there are 2 d.o.f. per node.

\[
\lim_{k \to \infty} \left( \frac{\text{d.o.f.}}{p} \right) = 4 \tag{32b}
\]

and therefore

\[
\lim_{k \to \infty} \left( \frac{\text{d.o.f.}}{\text{constraints}} \right) = \frac{4}{3} > 1 \tag{32c}
\]

For plane strain problems, when the mesh is uniformly refined, the QMITC-TLH element has a ratio (d.o.f./constraints) greater than one, therefore it does not lock.

#### 3.1.2. Axisymmetric analysis

\[
\text{tr}(\gamma_{i} \hat{H}) = \left[ \gamma_{ij} \left( \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} \right] + \left[ \gamma_{ii}^{(2)} \left( \frac{\partial}{\partial \epsilon} \hat{H}_{ii} \right|_{0}^{Dl} - \left. \frac{\partial}{\partial \epsilon} \hat{H}_{ii} \right|_{0}^{Dl} \right] J_{U} \gamma + \left[ \gamma_{ij}^{(2)} \left( \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} - \left. \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0}^{Dl} \right] J_{U} \gamma \tag{33}
\]

with \( i, j = 1, 2, 3 \) (\( \left. \frac{\partial}{\partial \epsilon} \hat{H}_{ij} \right|_{0} = 0 \), etc.).
Again, the fulfillment of Eq. (30) implies in the most general case three constraints. By doing the same analysis as before,

$$\lim_{k \to \infty} \left( \frac{\text{d.o.f.}}{\text{constraints}} \right) = \frac{4}{3} > 1. \quad (34)$$

For axisymmetric problems, when the mesh is uniformly refined, the QMITC-TLH element has a ratio (d.o.f./constraints) greater than one, therefore it does not lock.

4. Stress calculations

For the hyperelastic constitutive equation, we use the following free energy function in Eq. (23):

$$'\psi_e = \frac{1}{2} h' H_e : C : h' H_e , \quad (35)$$

where $C$ is a fourth order tensor. In the present implementation, we use a $C$-tensor isotropic and constant.

4.1. Radial return algorithm

In an incremental procedure for solving the non-linear problem once the nodal displacements corresponding to the configuration (or trial configuration during iterations) at $t + \Delta t$ are known, the stresses acting in that configuration are to be calculated.

Hence, for each element the data are:
- nodal incremental displacements;
- $\{ 'B, 'F_p, '\sigma_y \}$ for every Gauss point;
and we search for,
- $\{ ' + \Delta 'B, '0 + \Delta 'F_p, '0 + \Delta '\sigma_y \}$ for every Gauss point.

Therefore, at every Gauss point we go through the following calculation procedure:
- Using the displacements interpolation and Eqs. (29), calculate the total Hencky strain tensor $h' H$.

4.1.1. Elastic predictor

$$B^* = 'B , \quad (36a)$$

$$F_p^* = '0 F_p , \quad (36b)$$

$$\sigma_y^* = ' \sigma_y , \quad (36c)$$

$$0 + \Delta 'U = \exp('0 + \Delta 'H) \text{ (use Eqs. (2) and (4))} , \quad (36d)$$

$$0 + \Delta 'C = '0 + \Delta 'U \cdot '0 + \Delta 'U , \quad (36e)$$

$$C^*_e = (F_p^*)^{-1} \cdot '0 + \Delta 'C \cdot (F_p^*)^{-1} , \quad (36f)$$

$$H^*_e = \ln[(C^*_e)^{1/2}] , \quad (36g)$$

$$\Gamma^* = C : h' H^*_e , \quad (36h)$$

$$\phi^* = \left[ \frac{3}{2} (\Gamma^*_D - B^*) : (\Gamma^*_D - B^*) \right]^{1/2} - \sigma_y^* . \quad (36i)$$

If $\phi^* \leq 0$ then

$$' + \Delta 'B = B^* , \quad (37a)$$
\[ 1^{+ \Delta t} F_p = F_p^* , \]  
\[ 1^{+ \Delta t} \sigma_e = \sigma_e^* , \]  
\[ 1^{+ \Delta t} \Gamma = \Gamma^* . \]

else go to plastic corrector.

4.1.2. Plastic corrector:
(a) Plane strain and axisymmetric analysis. Since \( 1^{+ \Delta t} \omega_p = 0 \), the evolution equation for the intermediate configuration is
\[ 0^{+ \Delta t} \tilde{F}_p = \tilde{d}_p \cdot 0^{+ \Delta t} F_p . \]  
Following what has been done by Eterovic and Bathe [15], we perform a Euler backward integration
\[ 0^{+ \Delta t} F_p = \exp(\Delta t 1^{+ \Delta t} \tilde{d}_p) \cdot 0 F_p . \]  
\[ 0^{+ \Delta t} F_e = 0^{+ \Delta t} F - 0^{+ \Delta t} d_p \cdot \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) . \]  
Therefore
\[ 0^{+ \Delta t} F_e = F_e^* \cdot \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) , \]  
\[ 0^{+ \Delta t} C_e = \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) \cdot C_e^* \cdot \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) . \]  
where \( C_e^* = (F_e^*)^t \cdot (F_e^*) \). Also,
\[ 0^{+ \Delta t} C_e = \exp(2 0^{+ \Delta t} H_e) . \]  
From Eqs. (38d) and (38f), we obtain
\[ \exp(2 0^{+ \Delta t} H_e) = \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) \cdot \exp(2 H_e^*) \cdot \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) . \]  
Taking into account that the plastic deformation is isochoric when an associated Von Mises flow rule is used,
\[ |0^{+ \Delta t} F| = |F_e^*| = |0^{+ \Delta t} F_e| , \]  
and also
\[ \text{tr}(H_e^*) = \text{tr}(0^{+ \Delta t} H_e) . \]  
Therefore we can rewrite Eq. (38g) as
\[ \exp(2 0^{+ \Delta t} H_{e_D}) = \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) \cdot \exp(2 H_{e_D}^*) \cdot \exp(-\Delta t 1^{+ \Delta t} \tilde{d}_p) . \]  
where \( 0^{+ \Delta t} H_{e_D} \) and \( H_{e_D}^* \) are deviatoric tensors.

Since we are considering only elastically isotropic materials, the above leads to
\[ 0^{+ \Delta t} H_{e_D} = H_{e_D}^* - \Delta t 1^{+ \Delta t} \tilde{d}_p . \]  
For deviatoric components, the hyperelastic law in Eq. (35) implies
\[ 1^{+ \Delta t} \Gamma_D = 2G 0^{+ \Delta t} H_{e_D} , \]  
\[ \Gamma_D^* = 2G H_{e_D}^* . \]  
Hence,
\[ 1^{+ \Delta t} \Gamma_D = \Gamma_D^* - 2G \Delta t 1^{+ \Delta t} \tilde{d}_p . \]
Integrating the evolution Eq. (16b) with the Euler backward method, we obtain
\[ \dot{^\ast\ast}{\bf B} = \dot{\bf B} + \frac{2}{3} (1 - \beta) h \Delta t \dot{\bf \bar{d}}_p. \]  
(38o)

For the configuration at time \( t + \Delta t \),
\[ \dot{^\ast\ast}{\bf \Gamma}_D - \dot{^\ast\ast}{\bf B} = (\dot{\bf \Gamma}_D - \dot{\bf B}) - \frac{2}{3} \Delta t \dot{^\ast\ast}{\bf \bar{d}}_p (3G + (1 - \beta) h). \]  
(38p)

Using Eqs. (27) and (28), we obtain,
\[ (\dot{^\ast\ast}{\bf \Gamma}_D - \dot{^\ast\ast}{\bf B}) \left( 1 + \frac{(\dot{^\ast\ast}{\bf \bar{\varepsilon}}_p - \dot{\bf \bar{\varepsilon}}_p)[3G + (1 - \beta) h]}{\sqrt{\frac{3}{2} (\dot{^\ast\ast}{\bf \Gamma}_D - \dot{^\ast\ast}{\bf B}) : (\dot{^\ast\ast}{\bf \Gamma}_D - \dot{^\ast\ast}{\bf B})}} \right) = (\dot{\bf \Gamma}_D^* - \dot{\bf B}). \]  
(38q)

Let us call
\[ \dot{\bf \Gamma} = \sqrt{\frac{3}{2} (\dot{\bf \Gamma}_D - \dot{\bf B}) : (\dot{\bf \Gamma}_D - \dot{\bf B})} \]
equivalent Von Mises stress. Then
\[ \dot{\bf \Gamma} = \dot{\bf \Gamma}_* - (\dot{^\ast\ast}{\bf \bar{\varepsilon}}_p - \dot{\bf \bar{\varepsilon}}_p)[3G + 1 - \beta) h]. \]  
(38r)

The stresses at \( t + \Delta t \) have to satisfy the consistency condition \( \dot{^\ast\ast}{\bf \phi} = 0 \), hence
\[ \dot{\bf \Gamma}_* - (\dot{^\ast\ast}{\bf \bar{\varepsilon}}_p - \dot{\bf \bar{\varepsilon}}_p)[3G + (1 - \beta) h] = \dot{^\ast\ast}{\bf \sigma}_v = 0. \]  
(38s)

Using Eq. (16a), we obtain
\[ \dot{\bf \Gamma}_* - \dot{\bf \sigma}_v = \frac{\dot{\bf \bar{\varepsilon}}_p}{3G + h}, \]  
(39a)

where \( \dot{\bf \bar{\varepsilon}}_p = (\dot{^\ast\ast}{\bf \bar{\varepsilon}}_p - \dot{\bf \bar{\varepsilon}}_p) \).

Eq. (38q) shows the 'radial return property' of the plastic corrector algorithm, because
\[ \dot{^\ast\ast}{\bf \nu} = \dot{\bf \nu}_*. \]  
(39b)

Hence, from the above equations,
\[ \dot{^\ast\ast}{\bf \Gamma}_D = \dot{\bf \Gamma}_D^* - 2G \Delta t \dot{^\ast\ast}{\bf \bar{d}}_p, \]  
(39c)

and integrating with Euler's backward method,
\[ \dot{^\ast\ast}{\bf \Gamma}_D = \dot{\bf \Gamma}_D^* - \sqrt{6} G \Delta t \dot{\bf \bar{\varepsilon}}_p \dot{\bf \nu}_*. \]  
(39d)

With the above, we obtain \( \dot{^\ast\ast}{\bf \Gamma} \) remembering that in the case of the associated Von Mises rule, the hydrostatic stress is only related to the elastic deformations.

The back-stress tensor at \( t + \Delta t \) is
\[ \dot{^\ast\ast}{\bf B} = \dot{\bf B} + \frac{2}{3} (1 - \beta) h \Delta t \dot{\bf \bar{\varepsilon}}_p \dot{\bf \nu}_*, \]  
(39e)

and using Eq. (38b), we obtain
\[ \dot{^\ast\ast}{\bf F}_p = \exp \left( \sqrt{\frac{3}{2} \Delta \dot{\bf \bar{\varepsilon}}_p \dot{\bf \nu}_*} \right) \dot{\bf F}_p \]  
(39f)

and
\[ \dot{^\ast\ast}{\bf \sigma}_v = \dot{\bf \sigma}_v + \beta h \Delta \dot{\bf \bar{\varepsilon}}_p. \]  
(39g)
(b) Plane stress analysis. For this type of analysis, the plastic corrector algorithm presents some distinct features [34, 35].

In a plane stress problem in the \((x_1-x_2)\) plane, the \(x_3\) direction being normal to that plane
\[
T_{33}^{\text{pl}} = T_3^{\text{pl}} = 0.
\]

Let us call \(T_2\) the tensor formed by the \(T\) components acting in the \((x_1-x_2)\) plane. Obviously
\[
(T_2^{\text{D}})^{33} = -\text{tr}(T_2^{\text{D}}).
\]

with \(C_{\sigma}\) the elastic constitutive tensor for plane stress.

\[
T_2^{\text{D}} = C_{\sigma} : \varepsilon_2^\text{e}.
\]

In the above \(\varepsilon_2^\text{e}\) is formed by the components of the \(H_{\varepsilon}^\text{e}\) acting in the \((x_1-x_2)\) plane. Also,
\[
T_2^* = C_{\sigma} : \varepsilon_2^\text{e}.
\]

The deviatoric components are
\[
T_2^{\text{D}} = I_{\text{DEV}2} : \varepsilon_2^\text{e},
\]

\[
I_{\text{DEV}2}^{ijkl} = g^{ik} g^{jl} - \frac{1}{3} g^{ij} g^{kl}.
\]

After some algebra, we can show that

\[
T_2^{\text{D}} = A^{-1} : \left[ \begin{array}{c}
\Gamma_2^{\text{D}} + \frac{\sqrt{\frac{3}{2}} \Delta \varepsilon_p}{1 + \sqrt{\frac{2}{3}} (1 - \beta) h \Delta \varepsilon_p} \cdot C_{\sigma} : \mathbf{B} \end{array} \right].
\]

where

\[
A^{ijkl} = g^{ik} g^{jl} + C^{ijkl}_{\sigma} \frac{\sqrt{\frac{3}{2}} \Delta \varepsilon_p}{1 + \sqrt{\frac{2}{3}} (1 - \beta) h \Delta \varepsilon_p}.
\]

Replacing in the consistency condition \(T_2^{\text{D}} \phi = 0\), we obtain a non-linear equation for \(\Delta \varepsilon_p\) that is solved by any of the available methods (bisection, Newton, etc.).

Once \(\Delta \varepsilon_p\) is known, the calculation of \(T_2^{\text{D}}\) follows the same steps as in the previous case (radial return). The same comment is valid for the calculation of \(T_2^{\text{B}}, T_2^{\text{F}}\) and \(T_2^{\text{G}}\).

4.2. Recovery of Kirchhoff stresses

Once \(T_2^{\text{D}}\) is known, we are interested in recovering the Kirchhoff stresses \(\Delta \tau\):

\[
\Delta \tau = R_{\text{G}} \left( T_2^{\text{D}} \right).
\]

For calculating \(R_{\text{G}}\), we go through the following steps at each Gauss point:

- Calculate \(I_{\text{DEV}2}^{\text{D}}\) from the displacements interpolation and perform the polar decomposition to obtain \(I_{\text{DEV}2}^{\text{D!}}\).
- Using Eqs. (2) and (4), obtain \(I_{\text{DEV}2}^{\text{D!}} = \exp(I_{\text{DEV}2}^{\text{D}})\).
- Calculate \(I_{\text{DEV}2}^{\text{F}} = I_{\text{DEV}2}^{\text{D!}} \cdot I_{\text{DEV}2}^{\text{U}}\).
- Using the calculated \(I_{\text{DEV}2}^{\text{F}}\), calculate \(I_{\text{DEV}2}^{\text{E}} = I_{\text{DEV}2}^{\text{F}} \cdot I_{\text{DEV}2}^{\text{F}}^{-1}\).
- Perform the polar decomposition of \(I_{\text{DEV}2}^{\text{E}}\) to obtain \(R_{\text{G}}\).
5. The incremental formulation

If we assume a conservative loading and a fixed intermediate configuration, the equilibrium configuration at time (load level) \( t + \Delta t \) has to fulfil the Principle of Minimum Potential Energy \([25, 43]\):

\[
\delta^{i+\Delta t} \Pi(t^{i+\Delta t} H_e) = 0 ,
\]

where \( \delta^{i+\Delta t} \Pi \) is the potential energy corresponding to the \( t + \Delta t \) configuration.

By definition,

\[
i^{t+\Delta t} \Pi = \int_0^{t+\Delta t} \psi_e^0 \, dv - i^{t+\Delta t} \gamma ,
\]

where \( i^{t+\Delta t} \gamma \) is the potential of the external loads acting at \( t + \Delta t \).

We can therefore write, using Eqs. (45) and (46),

\[
\int_0^{t+\Delta t} \Gamma \cdot \delta^{t+\Delta t} H_e^0 \, dv = i^{t+\Delta t} P_i \cdot \delta u_i .
\]

In the above equation \( i^{t+\Delta t} P_i \) (\( i = 1, 2, \ldots, N \)) are the \( N \) external loads acting in the configuration at \( t + \Delta t \) and \( u_i \) are those loads displacements.

Since we are interpolating total Hencky strain components rather than elastic Hencky strain components, we have to use in Eq. (47)

\[
\delta^{t+\Delta t} H_e = \frac{\partial^{t+\Delta t} H_e}{\partial^{t+\Delta t} H} \cdot \delta^{t+\Delta t} H .
\]

The fourth order tensor \( i^{t+\Delta t} D = \frac{\partial^{t+\Delta t} H_e}{\partial^{t+\Delta t} H} \) is calculated at every Gauss point (see Appendix A). Note that \( D \) cannot be calculated at the sampling points (Fig. 3(b)) because the tensor \( i^{t+\Delta t} F_p \) is only known at Gauss points.

Replacing in Eq. (47) we obtain the set of non-linear equilibrium equations to be fulfilled by the configuration at \( t + \Delta t \):

\[
\int_0^{t+\Delta t} \Gamma \cdot \delta^{t+\Delta t} D^i \delta H_0^0 \, dv = i^{t+\Delta t} P_i \cdot \delta u_i .
\]

5.1. Linearization of the equilibrium equations

The non-linear equations (49) are solved using any available iterative method \([25, 44]\); therefore we linearize those equations around the \( t + \Delta t \) configuration obtained in the previous iteration. Therefore for the \( i \)th iteration,

\[
\int_0^{t+\Delta t} \left( \frac{\partial \Gamma}{\partial H} \bigg|_{t+\Delta t} \right) \delta^{i} D^i \delta H_0^0 \, dv \\
+ \int_0^{t+\Delta t} \Gamma^{i} \left( \frac{\partial}{\partial H} \delta^{i} D^i \delta H_0^0 \right)_{t+\Delta t} \, dv \\
= i^{t+\Delta t} P_i \cdot \delta u_i - \int_0^{t+\Delta t} \Gamma^{i} \delta^{i} D^i \delta H_0^0 \, dv .
\]

In the above

\[
\delta H = \frac{\partial H}{\partial C} \bigg|_{t+\Delta t} \delta C = 2 \frac{\partial H}{\partial C} \bigg|_{t+\Delta t} \delta \varepsilon ,
\]

where \( \delta \varepsilon \) is the variation of the Green–Lagrange strain tensor. The expression for this variation in terms of the displacements variation is given in \([25]\). For the derivation of the fourth order tensor \( \partial H / \partial C \), see Appendix A.
Also the tangent constitutive elasto-plastic tensor is

\[ t_{-\Delta t} C_{EP}^{\delta} = \frac{\partial \Gamma^{\delta}}{\partial H_e^{\delta}} |_{t_{-\Delta t}}. \]  

(50c)

If this fourth-order tensor is consistent with the radial return algorithm [6, 7, 34], the Newton iterative method will provide quadratic convergence [45].

It should be noted in the above that the resulting consistent tangent stiffness matrix is symmetric.

5.2. The algorithmic consistent tangent constitutive tensor

Based on previous developments by Simo et al. [34–36] we derive in this section a tangent constitutive tensor consistent with the radial return algorithm.

5.2.1. Plane strain and axisymmetric analysis

From Eqs. (15b), (26a) and (39c), we obtain

\[ \text{d} \Gamma_D = 2G [\text{d} H_e^D - \lambda ( \Gamma_D - 'B) - \lambda (\text{d} \Gamma_D - \text{d} B) ], \]  

(51a)

where we used \( \lambda = \frac{1}{2} \Delta \varepsilon_{\text{eq}} \Delta t / \sigma_{\text{eq}}. \)

From Eq. (39e),

\[ \text{d} B = \kappa_1 [\lambda ( \Gamma_D - 'B) + \lambda \text{d} \Gamma_D ] \]  

(51b)

and therefore,

\[ \text{d} \Gamma_D = \frac{2G}{1 + 2G \frac{\lambda}{\kappa_2}} \left[ \text{d} H_e^D - \frac{\lambda}{\kappa_2} ( \Gamma_D - 'B ) \right]. \]  

(51c)

In the above

\[ \kappa_1 = \frac{2}{3} (1 - \beta) \frac{h}{\kappa_2}, \]  

(51d)

\[ \kappa_2 = 1 + \frac{2}{3} (1 - \beta) h' \lambda. \]  

(51e)

Using Eqs. (15b), (16a) and (51c), we obtain

\[ \text{d} \lambda = \frac{9}{2} \kappa_2 \left(1 - \frac{2}{3} \beta h' \lambda\right) G \left[ \text{d} H_e^D - ( \Gamma_D - 'B ) : \text{d} H_e^D \right] \]  

(51f)

and replacing in Eq. (51c), we obtain

\[ \text{d} \Gamma_D = \frac{2G}{1 + 2G \frac{\lambda}{\kappa_2}} \left[ I - \kappa_3 ( \Gamma_D - 'B ) : ( \Gamma_D - 'B ) \right]: \text{d} H_e^D. \]  

(51g)

In the above \( I \) is the fourth-order unit symmetric tensor, and

\[ \kappa_3 = \frac{9 \left(1 - \frac{2}{3} \beta h' \lambda\right) G}{2 \sigma_{\text{eq}}^2 (3G + h' \kappa_2 + 2(1 - \beta)hG' \lambda)}. \]  

(51h)
Finally we can rewrite Eq. (51g) as

\[
d\Gamma_D = \frac{2G}{1 + 2G \frac{\kappa_A}{\kappa_2}} [I_{DEV} - \kappa_3 (\Gamma_D - \beta B)(\Gamma_D - \beta B)] : dH_e. \tag{51i}
\]

In the above

\[
I_{DEV} = I - \frac{1}{3} \tilde{g} \tilde{g}, \tag{51j}
\]

and \( \tilde{g} = R^e_s (g) \).

For a material behavior described by the associated Von Mises flow rule and a linear isotropic plastic relation,

\[
d\Gamma : \tilde{g} = \frac{E}{1 - 2\nu} \tilde{g} : dH_e. \tag{51k}
\]

Hence

\[
d\Gamma = \left\{ \frac{2G}{1 + 2G \frac{\kappa_A}{\kappa_2}} [I_{DEV} - \kappa_3 (\Gamma_D - \beta B)(\Gamma_D - \beta B)] + \frac{E}{3(1 - 2\nu)} \tilde{g} \tilde{g} \right\} : dH_e. \tag{51l}
\]

The fourth-order tensor relating \( dH_e \) with \( d\Gamma \) in Eq. (51l) is the algorithmic consistent tangent constitutive tensor \( C_{EP} \).

It is important to note that the tensor \( C_{EP} \) presents the following symmetries:

\[
C_{EP}^{ijkl} = C_{EP}^{jkl} = C_{EP}^{ijk} = C_{EP}^{kij} \tag{51m}
\]

5.2.2. Plane stress analysis

Operating as above but with the plane stress radial return algorithm, we obtain

\[
C_{EP} = \left[ A^{-1} : C_{\sigma} - \frac{(A^{-1} : C_{\sigma} : 'S)(A^{-1} : C_{\sigma} : 'S)}{(1 + \beta)('S : A^{-1} : C_{\sigma} : 'S)} \right] \tag{52}
\]

with \( 'S = (\Gamma_D - \beta B) \).

The above tensor also presents the symmetries described by Eq. (51m).

6. Numerical experimentation

In this section we present several numerical examples to demonstrate the performance of the QMITC-TLH finite strain elasto-plastic formulation.

6.1. Elasto-plastic cylinder under internal pressure

This example was also used in [22] to demonstrate the non-locking behavior of the QMITC element in infinitesimal strain elasto-plastic analysis.

In Fig. 4 we show two meshes used to analyze the deformation of an elasto-plastic cylinder under internal pressure. The solution is compared against a rigid-plastic analytical solution.

We see that the uniform and deformed meshes provide a very accurate solution. In Table 1 we show the quadratic convergence obtained using the Newton iteration technique.
6.2. Necking of a circular bar

The displacement controlled tension test of an elasto-perfectly plastic circular bar ($R_0 = 6.413; L_0 = 53.334$) is modelled using the QMITC-TLH meshes shown in Fig. 5(a). In order to localize the necking at the bar center, an initial imperfection is given (see Fig. 5(a)). In this case, the Newton method is combined with a line search procedure [25]. The deformed configuration corresponding to a

Table 1
Convergence of the elasto-plastic cylinder under internal pressure in the displacement and energy norms

<table>
<thead>
<tr>
<th>Step</th>
<th>Iteration</th>
<th>Undist. mesh*</th>
<th>Dist. mesh*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Unb. energy</td>
<td>Unb. energy</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ΔU</td>
</tr>
<tr>
<td>First</td>
<td>0</td>
<td>0.58E + 7</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.31E + 6</td>
<td>0.10E - 1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.11E + 5</td>
<td>0.58E - 2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.26E + 2</td>
<td>0.55E - 3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.19E - 3</td>
<td>0.12E - 4</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.11E - 13</td>
<td>0.21E - 3</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.85E - 6</td>
<td>0.21E - 10</td>
</tr>
<tr>
<td>Last</td>
<td>0</td>
<td>0.61E + 6</td>
<td>0.71E + 5</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.77E + 6</td>
<td>0.30E - 2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.23E - 1</td>
<td>0.13E - 3</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.21E - 10</td>
<td>0.90E - 7</td>
</tr>
</tbody>
</table>

*15 equal steps.

b 30 equal steps.
total elongation of 18% is shown in Fig. 5(a) for both meshes. In Fig. 5(b) we show the results for the tensile load obtained with both meshes as a function of the total elongation.

The QMITC-TLH results are compared with Bridgman empirical formulae [8], and we see a close agreement for the coarser mesh and a very good agreement for the finer mesh.

Finally in order to assess the effect of the initial imperfection on the results, we analyze the coarser model using a different initial imperfection: the complete section change is made across only one element. The tensile load results for both initial imperfections are compared in Fig. 5(c). It is evident that the result is not too sensitive to changes in the initial imperfection.

6.3. Axisymmetric sheet metal forming

The deep drawing with a spherical punch of an elasto-plastic circular plate fixed on its boundaries is analyzed using the QMITC-TLH mesh shown in Fig. 6(a). This case was also analyzed in [46−48], but using a different constitutive relation. The frictional contact problem is solved using the algorithm of [49, 50]. The deep drawing operation is analyzed for three different values of the Coulomb friction coefficient and the results are shown in Figs. 6(b)−(d).

6.4. Plane stress tension of a perforated plate

The case shown in Fig. 7(a) [34, 51] is analyzed using a 72 element mesh. In Figs. 7(b) and (c) we show the deformed configurations obtained using isotropic and kinematic hardening, respectively, for an axial elongation of 25%. Finally in fig. 7(d) we show the load−displacement relations obtained using both hardening hypotheses.
Fig. 6. Axisymmetric sheet metal forming.

Fig. 7. Plane stress tension of a perforated plate.
It is important to realize that the isotropic and kinematic hardening provide different solutions because since the principal stress directions rotate during the structural loading, there are local loading and unloading processes going on.

7. Conclusions

A finite element large strain elasto-plastic formulation based on the interpolation of displacements and covariant components of the total Hencky strain tensor was presented.

The formulation was developed using Lee’s multiplicative decomposition of the deformation gradient and the hyperelastic expression of Von Mises flow theory previously developed by Argyris and Dolsinis and by Simo and Ortiz.

The finite element formulation (QMITC-TLH) developed does not present spurious zero energy modes; satisfies Irons' patch test; does not lock in plane strain and axisymmetric analyses and presents a high predictive capability even when distorted meshes are used.

A symmetric consistent tangent stiffness matrix was also developed.

The simple numerical examples presented illustrate the performance of the QMITC-TLH formulation.

The development of more complex industrial applications requires the use of an efficient contact algorithm and adaptive meshing techniques [52].

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Appendix A: The 'D tensor

We defined in Section 5 the fourth order tensor

\[ 'D = \frac{\partial \sigma \mu H_e}{\partial \sigma} \]  

(A.1)

The eigenvalues of any symmetric second order tensor \( T \), with Cartesian components \( T_{ij} \) acting in a plane are

\[ \lambda_{1,2} = \frac{1}{2} [T_{11} + T_{22} \pm \sqrt{(T_{11} + T_{22})^2 + 4T_{12}^2}] \]  

(A.2a)

and the eigenvectors are

\[ \Phi_1 = [\alpha \quad \beta] \frac{1}{a} \]  

(A.2b)

\[ \Phi_2 = [-\beta \quad \alpha] \frac{1}{a} \]  

(A.2c)

If \( T_{11} > T_{22} \), then \( \alpha = 1, \beta = T_{12}/(T_{11} - \lambda_2) \) else \( \beta = 1, \alpha = T_{12}/(T_{22} - \lambda_2) \) and \( a^2 = \alpha^2 + \beta^2 \).

For the special case of \( T_{12} = 0 \) and \( T_{11} = T_{22} \), we introduce a small perturbation [15].

We can also write

\[ [T_{ij}] = \frac{1}{a^2} \begin{bmatrix} \alpha^2 \lambda_1 + \beta^2 \lambda_2 & \alpha \beta (\lambda_1 - \lambda_2) \\ \alpha \beta (\lambda_1 - \lambda_2) & \alpha^2 \lambda_2 + \beta^2 \lambda_1 \end{bmatrix} \]  

(A.3)

So using \( \lambda_H, \lambda_{H1}, \alpha_H \) and \( \beta_H \) corresponding to the Hencky strain tensor \( (\sigma H) \), we can write
\[ C = \frac{1}{\alpha_{\text{c}}^2} \left[ \alpha_{\text{c}}^2 \alpha_{\text{c}} + \beta_{\text{c}}^2 \alpha_{\text{c}} \right] \alpha_{\text{c}} \alpha_{\text{c}} \beta_{\text{c}} (\lambda_{\text{c}} - \lambda_{\text{c}}) + \beta_{\text{c}}^2 \alpha_{\text{c}} \]

with \( \lambda_{\text{c}} = \exp(2\lambda H_c) \).

Using Eq. (8), we obtain

\[ \dot{C} = F_p^{-1} \dot{\sigma} \text{CF}_p^{-1}, \]

and finally

\[ \dot{H}_c = \frac{1}{\alpha_{\text{c}}^2} \left[ \alpha_{\text{c}}^2 \lambda_{\text{c}} + \beta_{\text{c}}^2 \lambda_{\text{c}} \right] \alpha_{\text{c}} \alpha_{\text{c}} \beta_{\text{c}} (\lambda_{\text{c}} - \lambda_{\text{c}}) \]

with \( \lambda_{\text{c}} = \frac{1}{2} \ln(\alpha_{\text{c}}) \).

Hence using Eqs. (A.4) to (A.6), we can calculate

\[ \frac{\partial \dot{H}_c}{\partial \dot{C}_{\text{c}}} = \frac{\partial \dot{H}_h}{\partial \dot{C}_{\text{c}}} \frac{\partial \dot{C}_{\text{c}}}{\partial \dot{C}_{\text{c}}} \frac{\partial \dot{H}_h}{\partial \dot{C}_{\text{c}}} \frac{\partial \dot{C}_{\text{c}}}{\partial \dot{H}_h} \]

It is important to note that when calculating the derivatives in Eq. (A.7), \( \dot{F}_p \) is kept constant.

The second derivatives used in the tangent stiffness matrix are calculated in the same way.

References


