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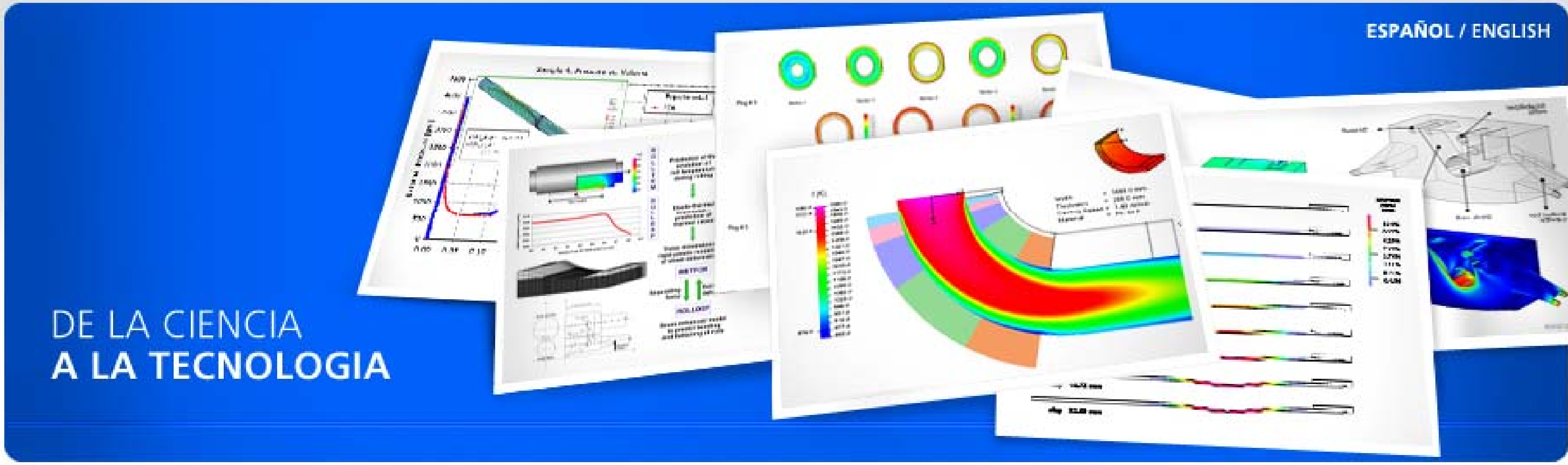

Advanced Topics in Computational Solid Mechanics. Industrial Applications

Section 3: Stress measures

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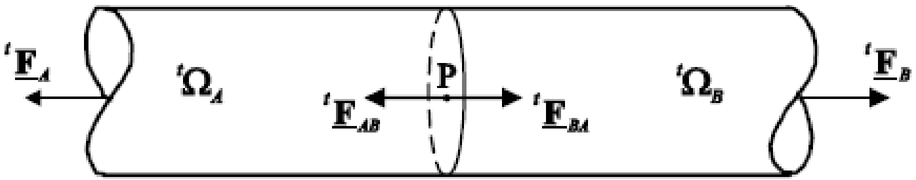
Stress Tensor

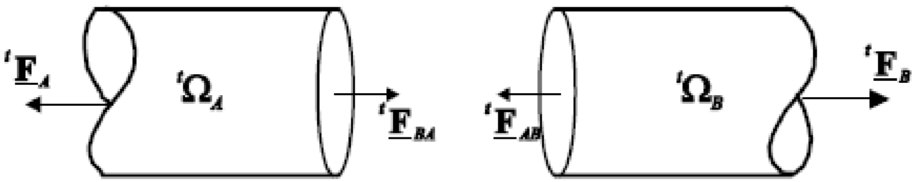
(a) $({}^t\Omega_A \cup {}^t\Omega_B)$ studied as a continuous bar

Body	${}^t\Omega_A \cup {}^t\Omega_B$
External forces	${}^t\mathbf{F}_A \,;\, {}^t\mathbf{F}_B$
Internal forces at P	${}^t\mathbf{F}_{AB}$ *(exerted by* ${}^t\Omega_A$ on ${}^t\Omega_B$)
	${}^t\mathbf{F}_{BA}$ *(exerted by* ${}^t\Omega_B$ on ${}^t\Omega_A$)

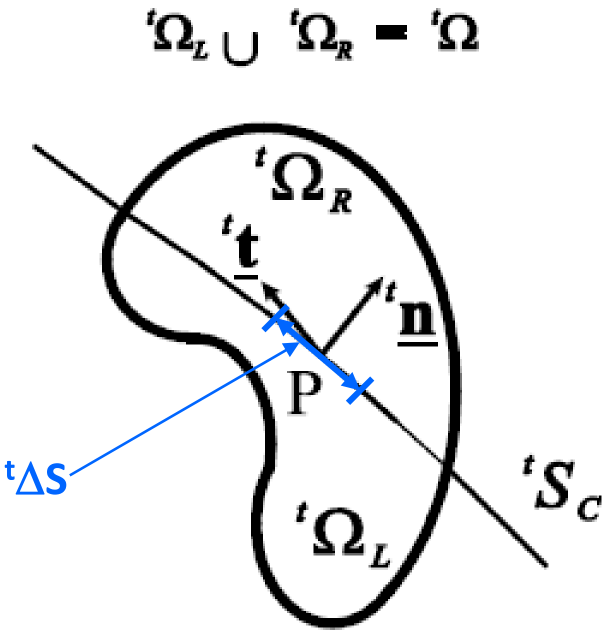


(b) ${}^t\Omega_A$ and ${}^t\Omega_B$ studied as free bodies

Bodies	${}^t\Omega_A \;;\; {}^t\Omega_B$
External forces	${}^t\mathbf{F}_A \,;\, {}^t\mathbf{F}_B \;;\; {}^t\mathbf{F}_{AB} \;;\; {}^t\mathbf{F}_{BA}$



The Cauchy Stress Tensor



$$\lim_{{}^t\Delta S \to 0} \frac{{}^t\Delta \underline{\mathbf{F}}}{{}^t\Delta S} = {}^t\underline{\mathbf{t}}$$

Non-polar media

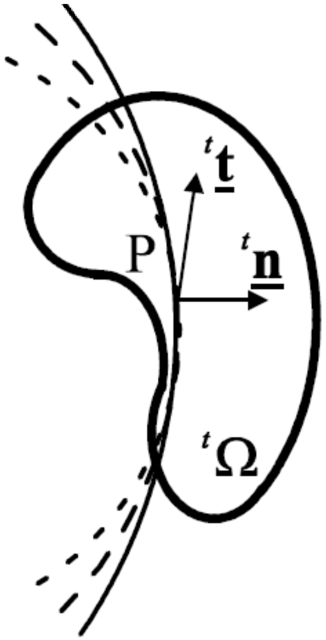
$$\lim_{{}^t\Delta S \to 0} \frac{{}^t\Delta \underline{\mathbf{M}}_P}{{}^t\Delta S} = \underline{\mathbf{0}}$$

Define

$${}^t\underline{\mathbf{t}} = {}^t\underline{\mathbf{n}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} \, .$$

${}^t\underline{\underline{\boldsymbol{\sigma}}}$, *the Cauchy stress tensor,*

The Cauchy Stress Tensor

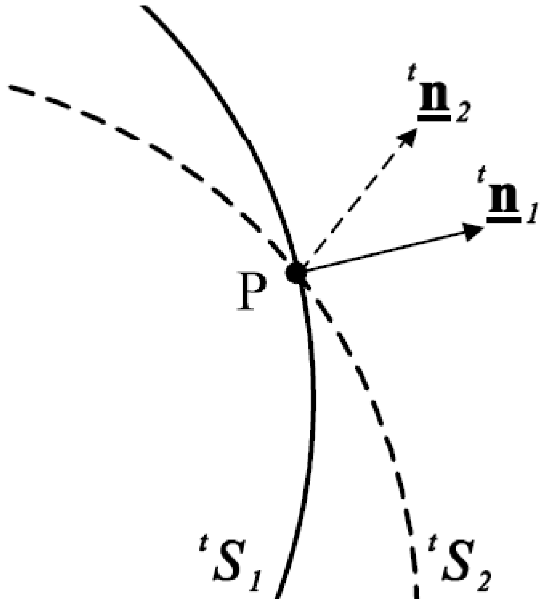


Tangent surfaces have the same traction vector

$$\boxed{{}^t\underline{\underline{\sigma}} = {}^t\underline{\underline{\sigma}}^T}$$

$$
\begin{aligned}
{}^t\underline{\mathbf{t}}_1 &= {}^t\underline{\mathbf{n}}_1 \cdot {}^t\underline{\underline{\sigma}} \\
{}^t\underline{\mathbf{t}}_2 &= {}^t\underline{\mathbf{n}}_2 \cdot {}^t\underline{\underline{\sigma}}
\end{aligned}
$$

$$\boxed{{}^t\underline{\mathbf{t}}_1 \cdot {}^t\underline{\mathbf{n}}_2 = {}^t\underline{\mathbf{t}}_2 \cdot {}^t\underline{\mathbf{n}}_1}$$



Conjugate stress/strain rate measures

Power provided by external forces

$$ {}^{t}P_{ext} = \int_{{}^{t}V} {}^{t}\rho \, {}^{t}\underline{\mathbf{b}} \cdot {}^{t}\underline{\mathbf{v}} \, {}^{t}\mathrm{d}V + \int_{{}^{t}S} {}^{t}\underline{\mathbf{t}} \cdot {}^{t}\underline{\mathbf{v}} \, {}^{t}\mathrm{d}S \, . $$

$$ {}^{t}P_{ext} = \int_{{}^{t}V} {}^{t}\rho \, {}^{t}\underline{\mathbf{b}} \cdot {}^{t}\underline{\mathbf{v}} \, {}^{t}\mathrm{d}V + \int_{{}^{t}S} \left({}^{t}\underline{\mathbf{n}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \right) \cdot {}^{t}\underline{\mathbf{v}} \, {}^{t}\mathrm{d}S $$

$$ {}^{t}P_{ext} = \int_{{}^{t}V} \left[{}^{t}\rho \, {}^{t}\underline{\mathbf{b}} \cdot {}^{t}\underline{\mathbf{v}} + \underline{\boldsymbol{\nabla}} \cdot \left({}^{t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t}\underline{\mathbf{v}} \right) \right] {}^{t}\mathrm{d}V $$

Conjugate stress/strain rate measures

$$\nabla \cdot \left({}^t\underline{\sigma} \cdot {}^t\underline{v} \right) = \frac{\partial}{\partial^t x}\left({}^t\sigma_{xx}\,{}^t v_x + {}^t\sigma_{xy}\,{}^t v_y + {}^t\sigma_{xz}\,{}^t v_z \right)$$

$$+ \frac{\partial}{\partial^t y}\left({}^t\sigma_{yx}\,{}^t v_x + {}^t\sigma_{yy}\,{}^t v_y + {}^t\sigma_{yz}\,{}^t v_z \right) + \cdots$$

$$= \left(\frac{\partial\, {}^t\sigma_{xx}}{\partial^t x} + \frac{\partial\, {}^t\sigma_{yx}}{\partial^t y} + \frac{\partial\, {}^t\sigma_{zx}}{\partial^t z} \right) {}^t v_x + \left(\frac{\partial\, {}^t\sigma_{xy}}{\partial^t x} + \frac{\partial\, {}^t\sigma_{yy}}{\partial^t y} + \frac{\partial\, {}^t\sigma_{zy}}{\partial^t z} \right) {}^t v_y + \cdots$$

$$+ {}^t\sigma_{xx} \frac{\partial^t v_x}{\partial^t x} + {}^t\sigma_{xy} \frac{\partial^t v_y}{\partial^t x} + {}^t\sigma_{xz} \frac{\partial^t v_z}{\partial^t x} + {}^t\sigma_{yx} \frac{\partial^t v_x}{\partial^t y} + {}^t\sigma_{yy} \frac{\partial^t v_y}{\partial^t y} + {}^t\sigma_{yz} \frac{\partial^t v_z}{\partial^t y} + \cdots$$

Conjugate stress/strain rate measures

$$\underline{\nabla}\cdot\left({}^{t}\underline{\underline{\sigma}}\cdot{}^{t}\underline{v}\right)={}^{t}\sigma_{ij}\frac{\partial\,{}^{t}v_{i}}{\partial\,{}^{t}x_{j}}+\underline{\nabla}\cdot{}^{t}\underline{\underline{\sigma}}\cdot{}^{t}\underline{v}={}^{t}\underline{\underline{\sigma}}\cdot{}^{t}\underline{\underline{l}}+\left(\underline{\nabla}\cdot{}^{t}\underline{\underline{\sigma}}\right)\cdot{}^{t}\underline{v}$$

Since ${}^{t}\underline{\underline{\sigma}}$ is symmetric

$$\boxed{{}^{t}\underline{\underline{\sigma}}\cdot{}^{t}\underline{\underline{l}}={}^{t}\underline{\underline{\sigma}}\cdot{}^{t}\underline{\underline{d}}}$$

$$\underline{\nabla}\cdot\left({}^{t}\underline{\underline{\sigma}}\cdot{}^{t}\underline{v}\right)={}^{t}\underline{\underline{\sigma}}\cdot{}^{t}\underline{\underline{d}}+\left(\underline{\nabla}\cdot{}^{t}\underline{\underline{\sigma}}\right)\cdot{}^{t}\underline{v}$$

Conjugate stress/strain rate measures

$$\boxed{{}^t\rho\,{}^t\underline{\mathbf{b}} + \underline{\boldsymbol{\nabla}} \cdot {}^t\underline{\underline{\boldsymbol{\sigma}}} = {}^t\rho\,\frac{D\,{}^t\underline{\mathbf{v}}}{Dt}}$$

$${}^tP_{ext} = \int_{{}^tV} \left[\, {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}} + {}^t\rho\,\frac{D\,{}^t\underline{\mathbf{v}}}{Dt} \cdot {}^t\underline{\mathbf{v}} \,\right] {}^t\mathrm{d}V$$

The kinetic energy is,

$${}^tK = \int_{{}^tV} \frac{{}^t\rho}{2}\,{}^t\underline{\mathbf{v}} \cdot {}^t\underline{\mathbf{v}}\,{}^t\mathrm{d}V = \int_m \frac{1}{2}\,{}^t\underline{\mathbf{v}} \cdot {}^t\underline{\mathbf{v}}\,\mathrm{d}m$$

$$\frac{D\,{}^tK}{Dt} = \int_m \frac{D\,{}^t\underline{\mathbf{v}}}{Dt} \cdot {}^t\underline{\mathbf{v}}\,\mathrm{d}m\,.$$

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Conjugate stress/strain rate measures

$$
{}^{t}P_{ext} = \frac{D^{t}K}{Dt} + \int_{{}^{t}V} {}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\mathbf{d}}}\ {}^{t}\mathrm{d}V
$$

We define

$$
{}^{t}P_{\sigma} = \int_{{}^{t}V} {}^{t}\underline{\underline{\boldsymbol{\sigma}}} : {}^{t}\underline{\underline{\mathbf{d}}}\ {}^{t}\mathrm{d}V
$$

${}^{t}P_{\sigma}$: stored or dissipated by the material

$\boxed{{}^{t}\underline{\underline{\boldsymbol{\sigma}}} \text{ and } {}^{t}\underline{\underline{\mathbf{d}}} \text{ are energy conjugate}}$

Conjugate stress/strain rate measures

Kirchhoff stress tensor

$$ {}^tP_\sigma = \int_{{}^tV} {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}}\, {}^t\mathrm{d}V = \int_{{}^\circ V} \frac{{}^\circ\rho}{{}^t\rho}\, {}^t\underline{\underline{\boldsymbol{\sigma}}} : {}^t\underline{\underline{\mathbf{d}}}\, {}^\circ\mathrm{d}V \,. $$

The *Kirchhoff stress tensor* is defined as

$$ {}^t\underline{\underline{\boldsymbol{\tau}}} = \frac{{}^\circ\rho}{{}^t\rho}\, {}^t\underline{\underline{\boldsymbol{\sigma}}} $$

$\boxed{{}^t\underline{\underline{\boldsymbol{\tau}}} \text{ and } {}^t\underline{\underline{\mathbf{d}}} \text{ are energy conjugate}}$

$${}^t\underline{\underline{\boldsymbol{\tau}}} \left\{ \begin{array}{l} \text{Eulerian tensor} \\ \text{Symmetric tensor} \end{array} \right.$$

Conjugate stress/strain rate measures

First Piola-Kirchhoff stress tensor

Define $\boxed{{}_o^tP_{Ij} = {}^t\tau_{pj}\left({}_o^tX^{-1}\right)_{Ip}}$; ${}_o^t\underline{\underline{P}} = {}_o^tP_{Ij}\ {}^o\underline{e}_I\ {}^t\underline{e}_j$; ${}_o^t\dot{\underline{\underline{X}}} = {}^t\underline{l}\cdot{}_o^t\underline{\underline{X}}$

$$\int_{{}^oV} {}_o^tP_{Ij}\ {}^o\underline{e}_I\ {}^t\underline{e}_j \cdot\cdot\ {}_o^t\dot{X}_{mM}\ {}^t\underline{e}_m\ {}^o\underline{e}_M\ {}^o dV =$$

$$\int_{{}^oV} {}_o^tP_{Ij}\ {}^tl_{mn}\ {}_o^tX_{nM}\ \delta_{jm}\ \delta_{IM}\ {}^o dV =$$

$$\int_{{}^oV} {}_o^tP_{Ij}\ {}^tl_{jn}\ \frac{\partial\,{}^tx_n}{\partial\,{}^ox_I}\ {}^o dV =$$

$$\int_{{}^oV} {}_o^t\tau_{pj}\ \frac{\partial\,{}^ox_I}{\partial\,{}^tx_p}\ {}^tl_{jn}\ \frac{\partial\,{}^tx_n}{\partial\,{}^ox_I}\ {}^o dV =$$

$$\delta_{np}$$

Conjugate stress/strain rate measures

First Piola-Kirchhoff stress tensor

$$\int_{{}^oV} {}^t_o\underline{\underline{P}} \cdot\cdot\, {}^t_o\underline{\underline{X}} \; {}^o dV = \int_{{}^oV} {}^t\tau_{pl} \; {}^t l_{jp} \; {}^o dV$$

$$\int_{{}^oV} {}^t\underline{\underline{\tau}} : {}^t\underline{\underline{l}} \; {}^o dV = \int_{{}^oV} {}^t\underline{\underline{\tau}} : {}^t\underline{\underline{d}} \; {}^o dV$$

$\boxed{{}^t_o\underline{\underline{P}} \text{ and } {}^t_o\underline{\underline{X}} \text{ are energy conjugate}}$

Conjugate stress/strain rate measures

Second Piola-Kirchhoff stress tensor

Define

$$\boxed{{}_{o}^{t}\underline{\underline{S}} = {}_{o}^{t}S_{AB}\ {}^{o}\underline{e}_{A}\ {}^{o}\underline{e}_{B}}$$

$${}_{o}^{t}S_{AB} = {}^{t}\tau_{ab}\left({}_{o}^{t}X^{-1}\right)_{Aa}\left({}_{o}^{t}X^{-1}\right)_{Bb}$$

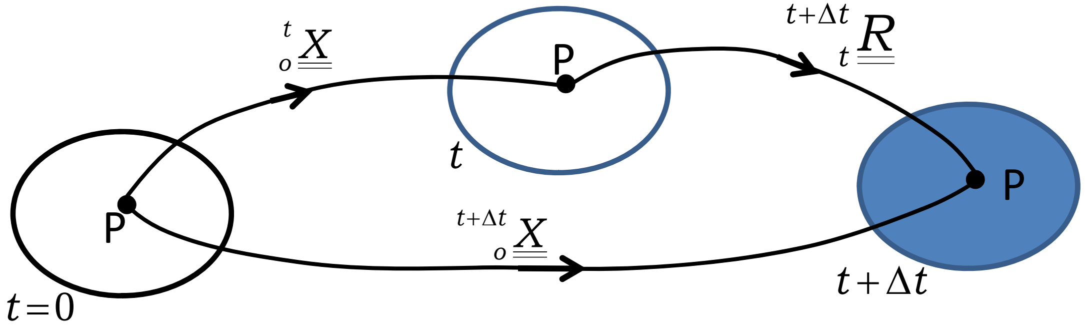
and demonstrate that $$\int_{{}^{o}V} {}_{o}^{t}S_{AB}\ {}_{o}^{t}\dot{\varepsilon}_{AB}\ {}^{o}dV = \int_{{}^{o}V} {}^{t}\tau_{ab}\ {}^{t}d_{ab}\ {}^{o}dV$$

$${}_{o}^{t}\underline{\underline{S}}$$ and $${}_{o}^{t}\dot{\underline{\underline{\varepsilon}}}$$ are energy conjugate

${}_{o}^{t}\underline{\underline{S}}$
- Lagrangean tensor
- Symmetric tensor

Conjugate stress/strain rate measures

Second Piola-Kirchhoff stress tensor



At the point P we can write: $$^{t+\Delta t}_{o}\underline{\underline{X}} = {}^{t+\Delta t}_{t}\underline{\underline{R}} \cdot {}^{t}_{o}\underline{\underline{X}}$$

and therefore, for any vector (e.g. $^{t}\underline{v}$, $^{t}\underline{n}$)

$$^{t+\Delta t}\underline{\mathbf{r}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{r}}$$

Conjugate stress/strain rate measures

Second Piola-Kirchhoff stress tensor

Example 3.3. ◀◀◀◀◀

For an arbitrary force vector ${}^{t}\underline{\mathbf{f}}$ (it can be a force per unit surface, per unit volume, etc.) and considering the evolution described above,

$$\begin{aligned} {}^{t+\Delta t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} &= {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} \\ &= {}^{t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} \\ &= {}^{t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{v}} \end{aligned}$$

and since the rotation tensor is orthogonal,

$${}^{t+\Delta t}\underline{\mathbf{f}} \cdot {}^{t+\Delta t}\underline{\mathbf{v}} = {}^{t}\underline{\mathbf{f}} \cdot {}^{t}\underline{\mathbf{v}} \, .$$

The above equation states the intuitive notion that a rigid-body rotation cannot affect the value of the deformation power performed by the external forces. ◀◀◀◀◀

Conjugate stress/strain rate measures

Second Piola-Kirchhoff stress tensor

At t we can write $\quad {}^{t}\underline{\mathbf{t}} = {}^{t}\underline{\mathbf{n}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}}$

and at $(t+\Delta t)$, $\quad {}^{t+\Delta t}\underline{\mathbf{t}} = {}^{t+\Delta t}\underline{\mathbf{n}} \cdot {}^{t+\Delta t}\underline{\underline{\boldsymbol{\sigma}}}$

$$\left.\begin{aligned} {}^{t+\Delta t}\underline{\mathbf{n}} &= {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{n}} \\ {}^{t+\Delta t}\underline{\mathbf{t}} &= {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\mathbf{t}} \end{aligned}\right\}$$

$${}^{t+\Delta t}_{t}\underline{\underline{R}} \cdot {}^{t}\underline{t} = {}^{t+\Delta t}_{t}\underline{\underline{R}} \cdot {}^{t}\underline{n} \cdot {}^{t+\Delta t}\underline{\underline{\sigma}}$$

$${}^{t+\Delta t}_{t}R_{ab}\, {}^{t}t_{b} = {}^{t+\Delta t}_{t}R_{lm}\, {}^{t}n_{m}\, {}^{t+\Delta t}\sigma_{la} = {}^{t}n_{m}\, {}^{t+\Delta t}_{t}R_{ml}^{T}\, {}^{t+\Delta t}\sigma_{la}$$

$$\underbrace{{}^{t+\Delta t}_{t}R_{pa}^{T}\, {}^{t+\Delta t}_{t}R_{ab}}_{\delta_{pb}}\, {}^{t}t_{b} = {}^{t}n_{m}\, {}^{t+\Delta t}_{t}R_{ml}^{T}\, {}^{t+\Delta t}\sigma_{la}\, {}^{t+\Delta t}_{t}R_{pa}^{T}$$

$${}^{t}t_{p} = {}^{t}n_{m}\, \underbrace{{}^{t+\Delta t}_{t}R_{ml}^{T}\, {}^{t+\Delta t}\sigma_{la}\, {}^{t+\Delta t}_{t}R_{ap}}_{{}^{t}\sigma_{mp}}$$

Conjugate stress/strain rate measures

Second Piola-Kirchhoff stress tensor

$$ {}^{t}\underline{\underline{\boldsymbol{\sigma}}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} \; {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} $$

$$ {}^{t+\Delta t}\underline{\underline{\boldsymbol{\sigma}}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} \cdot {}^{t}\underline{\underline{\boldsymbol{\sigma}}} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}}^{T} $$

$$ {}^{t+\Delta t}_{t}\underline{\underline{S}} = \frac{{}^{o}\rho}{{}^{t}\rho}\, {}^{t+\Delta t}_{o}\underline{\underline{X}}^{-1} \cdot {}^{t+\Delta t}\underline{\underline{\sigma}} \cdot {}^{t+\Delta t}_{o}\underline{\underline{X}}^{-T} = \frac{{}^{o}\rho}{{}^{t}\rho}\, {}^{t}_{o}\underline{\underline{X}}^{-1} \cdot \underbrace{{}^{t+\Delta t}_{o}\underline{\underline{R}}^{T} \cdot {}^{t+\Delta t}\underline{\underline{\sigma}} \cdot {}^{t+\Delta t}_{o}\underline{\underline{R}}}_{{}^{t}\underline{\underline{\sigma}}} \cdot {}^{t+\Delta t}_{o}\underline{\underline{X}}^{-T} $$

$$ \boxed{ {}^{t+\Delta t}_{\circ}\underline{\underline{\mathbf{S}}} = {}^{t}_{\circ}\underline{\underline{\mathbf{S}}} \qquad \text{when} \quad {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{X}}} = {}^{t+\Delta t}_{t}\underline{\underline{\mathbf{R}}} } $$

Conjugate stress/strain rate measures

Hencky stress tensor

$$ {}^{t+\Delta t}_{\ \ t}\underline{\underline{H}} = \ln\left({}^{t+\Delta t}_{\ \ t}\underline{\underline{U}} \right) $$

Define the symmetric tensor $\quad {}^{t}\Gamma_{AB} = \left({}^{t+\Delta t}_{\ \ t}R^{T} \right)_{Aa} \; {}^{t}\tau_{ab} \left({}^{t+\Delta t}_{\ \ t}R \right)_{bB}$

It is shown that only for isotropic material

$$ \boxed{ {}^{t}P_{\sigma}^{Isot.Mat.} = \int_{{}^{\circ}V} {}^{t}\underline{\underline{\mathbf{\Gamma}}} : {}^{t}_{\circ}\dot{\underline{\underline{\mathbf{H}}}} \; {}^{\circ}\mathrm{d}V } $$