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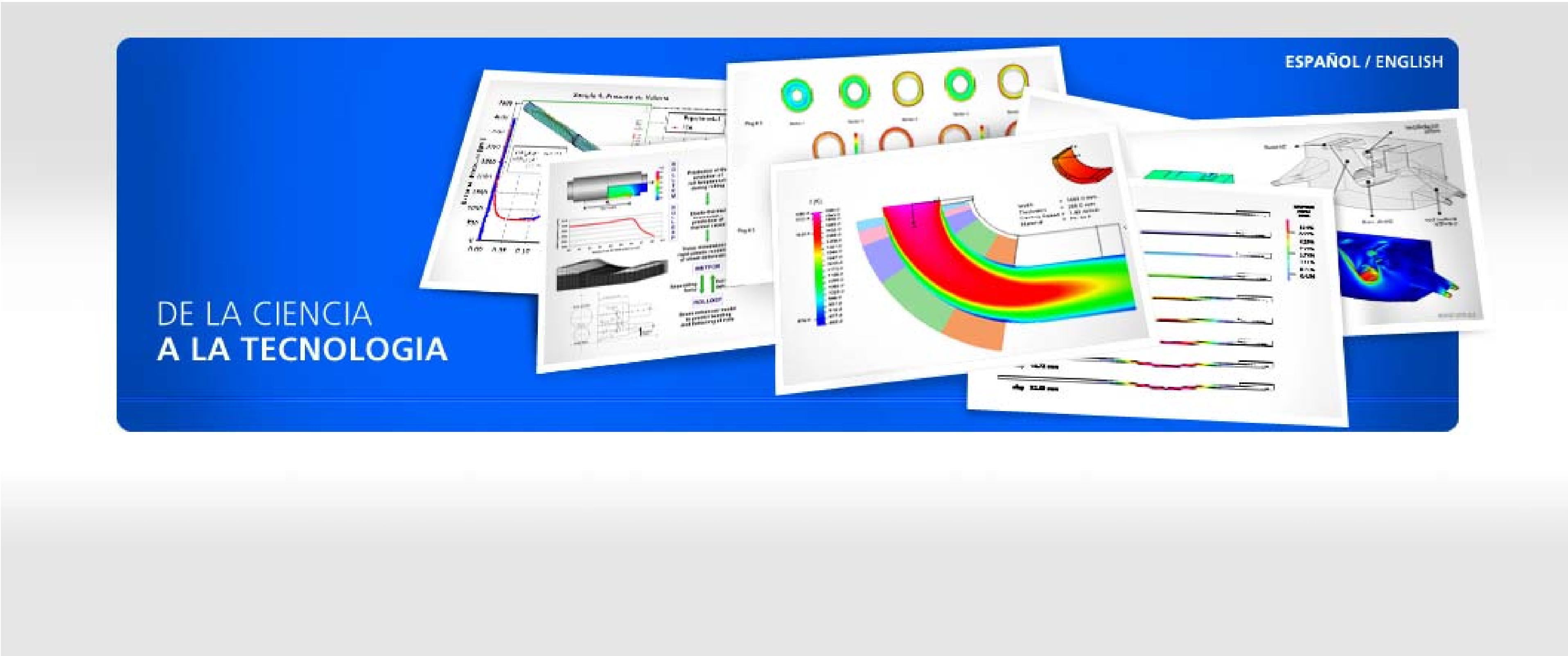

Advanced Topics in Computational Solid Mechanics.

Section 4: The Principle of Virtual Work. Linear Formulation and Incremental Formulations for Nonlinear Analyses

Eduardo N. Dvorkin

Stanford University
Mechanical Engineering
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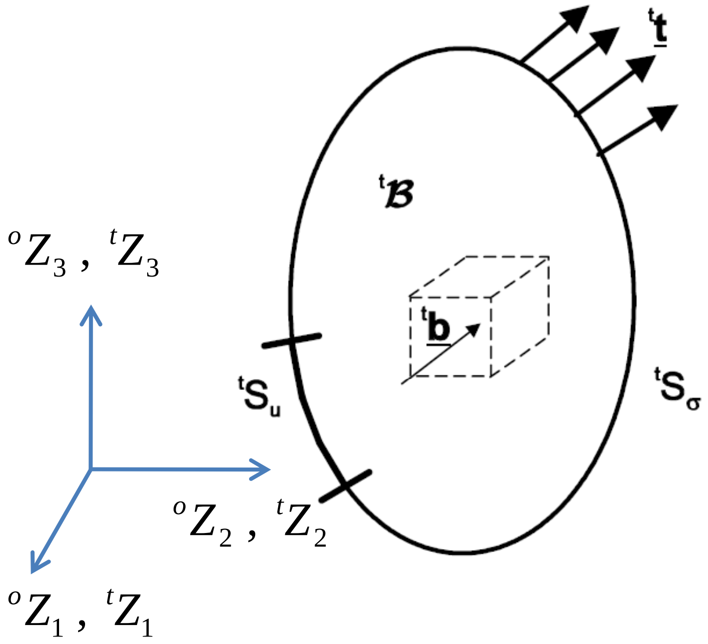
The Principle of Virtual Work



$$ {}^tS = {}^tS_u \cup {}^tS_\sigma $$
$$ {}^tS_u \cap {}^tS_\sigma = \varnothing \; . $$

We can also define an *admissible displacements field* as (Fung 1965),

$$ {}^t\underline{\tilde{\mathbf{u}}}({}^tz_\alpha) = {}^t\underline{\mathbf{u}}({}^tz_\alpha) + \delta {}^t\underline{\mathbf{u}}({}^tz_\alpha) \; . $$

$\delta {}^t\underline{\mathbf{u}} \equiv \underline{0}$ on ${}^tS_u$ and they are arbitrary on ${}^tS_\sigma$,

The Principle of Virtual Work

Assuming that when the continuum evolves from ${}^t\underline{\mathbf{u}}$ to ${}^t\underline{\widetilde{\mathbf{u}}}$, the external loads remain constant, the work performed by them is the *virtual work of the external loads* ($\delta^t W_{ext}$),

$$\delta^t W_{ext} = \int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\ {}^t\rho\ {}^t\mathrm{d}V + \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\ {}^t\mathrm{d}S\ . \tag{6.2}$$

$$\begin{aligned}\int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\ {}^t\mathrm{d}S &= \int_{{}^tS_\sigma} \left({}^t\sigma_{\alpha\beta}\ \delta^t u_\beta\right)\ {}^tn_\alpha\ {}^t\mathrm{d}S \\ &= \int_{{}^tS} \left({}^t\sigma_{\alpha\beta}\ \delta^t u_\beta\right)\ {}^tn_\alpha\ {}^t\mathrm{d}S \\ &= \int_{{}^tV} \left({}^t\sigma_{\alpha\beta}\ \delta^t u_\beta\right)_{,\alpha}\ {}^t\mathrm{d}V\ .\end{aligned}$$

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$$\int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta {}^t\underline{\mathbf{u}}\ {}^t\mathrm{d}S = \int_{{}^tV} {}^t\sigma_{\alpha\beta,\alpha}\ \delta {}^tu_\beta\ {}^t\mathrm{d}V + \int_{{}^tV} {}^t\sigma_{\alpha\beta}\ \delta({}^tu_{\beta,\alpha})\ {}^t\mathrm{d}V$$

In the last integral we have used the equality (Fung 1965)

$$\delta\left(\frac{\partial {}^tu_\beta}{\partial {}^tz_\alpha}\right) = \frac{\partial}{\partial {}^tz_\alpha}\left(\delta {}^tu_\beta\right)\ .$$

$$\delta {}^tW_{ext} = \int_{{}^tV} {}^t\rho\ \ {}^tb_\beta\ \delta {}^tu_\beta\ \ {}^tdV\ +$$

$$\int_{{}^tV} {}^t\sigma_{\alpha\beta,\alpha}\ \delta {}^tu_\beta\ {}^t\mathrm{d}V + \int_{{}^tV} {}^t\sigma_{\alpha\beta}\ \delta({}^tu_{\beta,\alpha})\ {}^t\mathrm{d}V$$

The Principle of Virtual Work

Notes:

✓ From internal equilibrium

$$ {}^{t}\sigma_{\alpha\beta,\alpha} + {}^{t}\rho \; {}^{t}b_{\beta} = o $$

(in dynamic problems we include in ${}^{t}\underline{\mathbf{b}}$ the inertia forces)

$$ \delta\,{}^{t}W_{ext} = \int_{{}^{t}V} {}^{t}\sigma_{\alpha\beta} \;\; \delta\left({}^{t}u_{\beta,\alpha}\right) {}^{t}dV $$

The Principle of Virtual Work

Infinitesimal strain tensor components

$$ {}^{t}\varepsilon_{\alpha\beta} = \frac{1}{2}\left({}^{t}u_{\alpha,\beta} + {}^{t}u_{\beta,\alpha} \right) $$

Virtual strains

$$ \delta^{t}\varepsilon_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial\left(\delta^{t}u_{\alpha}\right)}{\partial^{t}z_{\beta}} + \frac{\partial\left(\delta^{t}u_{\beta}\right)}{\partial^{t}z_{\alpha}} \right] $$

$$ {}^{t}\sigma_{\alpha\beta}\,\delta^{t}u_{\beta,\alpha} = {}^{t}\sigma_{\alpha\beta}\,\delta^{t}\varepsilon_{\alpha\beta} $$

$$ \delta^{t}W_{ext} = \underbrace{\int_{{}^{t}V} {}^{t}\sigma_{\alpha\beta}\;\delta\left({}^{t}\varepsilon_{\alpha\beta}\right)\,{}^{t}dV}_{\delta^{t}W_{\text{int}}} $$

The Principle of Virtual Work

Notes

- No assumption was made on the material, i.e. on its stress - strain relation; hence, the Principle of Virtual Work holds for any constitutive relation.
- No assumption was made on the actual strains in the spatial configuration; hence, the Principle of Virtual Work holds for finite or infinitesimal strains.
- No assumption was made on the external loads; hence, the Principle of Virtual Work holds for conservative and nonconservative loads (Crandall 1956).
- The integrals are calculated on the spatial configuration of the body.
- The Principle of Virtual Work was derived from momentum conservation **and not** from energy conservation.

The Principle of Virtual Work

Equilibrium → PVW

PVW → Equilibrium

Geometrically linear analysis

$$ {}^{t}V \cong {}^{\circ}V $$

$$ {}^{t}S \cong {}^{\circ}S $$

$$ \int_{{}^{\circ}V} {}^{t}\underline{\mathbf{b}} \cdot \delta {}^{t}\underline{\mathbf{u}} \, {}^{o}\rho \, {}^{\circ}\mathrm{d}V + \int_{{}^{\circ}S_{\sigma}} {}^{t}\underline{\mathbf{t}} \cdot \delta {}^{t}\underline{\mathbf{u}} \, {}^{o}\mathrm{d}S = \int_{{}^{o}V} {}^{t}\underline{\underline{\boldsymbol{\sigma}}} : \delta {}^{t}\underline{\underline{\boldsymbol{\varepsilon}}} \, {}^{o}\mathrm{d}V $$

The Principle of Virtual Work

Geometrically non-linear analysis

$$\delta^t \varepsilon_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial\left(\delta^t u_\alpha\right)}{\partial^t z_\beta} + \frac{\partial\left(\delta^t u_\beta\right)}{\partial^t z_\alpha}\right]$$

$$\frac{\mathrm{d}\delta^t \varepsilon_{\alpha\beta}}{\mathrm{d}t} = \frac{1}{2}\left(\frac{\partial\delta^t v_\alpha}{\partial^t z_\beta} + \frac{\partial\delta^t v_\beta}{\partial^t z_\alpha}\right) = \delta\left({}^t d_{\alpha\beta}\right)$$

where $\delta^t \underline{\mathbf{v}}$ is the virtual velocity vector.

The Principle of Virtual Work

$$\int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\ {}^t\rho\ {}^t\mathrm{d}V + \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\ {}^t\mathrm{d}S = \int_{{}^\circ V} {}^t\underline{\underline{\boldsymbol{\tau}}} : \delta^t\underline{\underline{\boldsymbol{\varepsilon}}}\ {}^\circ\mathrm{d}V$$

From energy conjugate sets

$$ {}_o^tS_{AB} \frac{d\,{}_o^t\varepsilon_{AB}}{dt} = {}^t\tau_{ab} \frac{d\,{}^t\varepsilon_{ab}}{dt}$$

$$ {}_o^tS_{AB}\, d\,{}_o^t\varepsilon_{AB} = {}^t\tau_{ab}\, d\,{}^t\varepsilon_{ab}$$

$$ {}_o^tS_{AB}\, \delta\,{}_o^t\varepsilon_{AB} = {}^t\tau_{ab}\, \delta\,{}^t\varepsilon_{ab}$$

$$\int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta^t\underline{\mathbf{u}}\ {}^t\rho\ {}^t\mathrm{d}V + \int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta^t\underline{\mathbf{u}}\ {}^t\mathrm{d}S = \int_{{}^\circ V} {}_o^t\underline{\underline{\mathbf{S}}} : \boldsymbol{\delta}\ {}_o^t\underline{\underline{\boldsymbol{\varepsilon}}}\ {}^\circ\mathrm{d}V$$

The Principle of Virtual Work

General load per unit mass or per unit surface

$$ {}^{t}\underline{\mathbf{f}} = {}^{l}\lambda\ {}^{m}\underline{\varphi}\ . \qquad (6.14)$$

In the above, ${}^{l}\lambda$ is a scalar proportional to the load modulus and (Schweizerhof & Ramm 1984):

- $l = 0$ implies that the load modulus is a function of the reference coordinates *(body-attached loads)*;
- $l = t$ implies that the load modulus is a function of the spatial coordinates *(space-attached loads)*.

The unitary vector ${}^{m}\underline{\varphi}$ is a direction and (Schweizerhof & Ramm 1984):

- $m = 0$ implies a constant direction load;
- $m = t$ implies a follower load (the direction is a function of the body displacements).

The Principle of Virtual Work

Example 6.1. ◄◄◄◄◄

Buckling of a circular ring.

In (Brush & Almroth 1975) we find that the elastic buckling pressure acting on a circular ring depends on the type of load that we consider:

Load	*Buckling pressure*
Hydrostatic pressure loading	$p_{cr} = 3\frac{EI}{a^3}$
Centrally directed pressure loading	$p_{cr} = 4.5\frac{EI}{a^3}$

Both are cases of follower loads but, the description of the load as a function of the displacements is different. ◄◄◄◄◄

The Principle of Virtual Work

$$\int_{{}^tV} {}^t\underline{\mathbf{b}} \cdot \delta{}^t\underline{\mathbf{u}}\; {}^t\rho\; {}^t\mathrm{d}V = \int_{{}^\circ V} {}^t\underline{\mathbf{b}} \cdot \delta{}^t\underline{\mathbf{u}}\; {}^o\rho\; {}^\circ\mathrm{d}V$$

$$\int_{{}^tS_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta{}^t\underline{\mathbf{u}}\; {}^t\mathrm{d}S = \int_{{}^\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta{}^t\underline{\mathbf{u}}\; {}^tJ_S\; {}^\circ\mathrm{d}S$$

PVW

$$\int_{{}^\circ V} {}^t\underline{\mathbf{b}} \cdot \delta{}^t\underline{\mathbf{u}}\; {}^o\rho\; {}^\circ\mathrm{d}V + \int_{{}^\circ S_\sigma} {}^t\underline{\mathbf{t}} \cdot \delta{}^t\underline{\mathbf{u}}\; {}^tJ_S\; {}^\circ\mathrm{d}S = \int_{{}^\circ V} {}^t_o\underline{\underline{\mathbf{S}}} : \boldsymbol{\delta}\, {}^t_o\underline{\underline{\boldsymbol{\varepsilon}}}\; {}^\circ\mathrm{d}V$$

The Principle of Virtual Work

Example 6.4. ◀◀◀◀◀

Stability of the equilibrium configuration (buckling) (Hoff 1956).

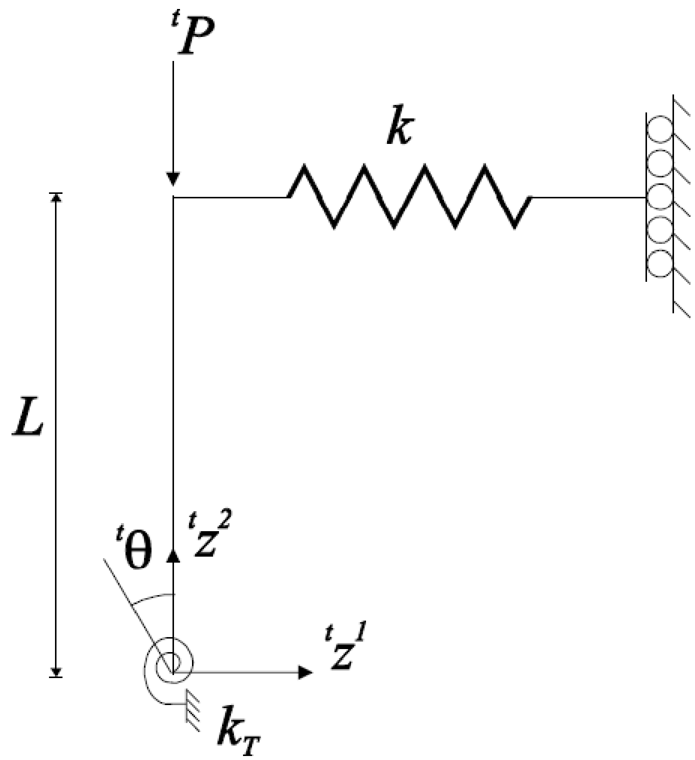
Let us consider the system shown in the following figure, in equilibrium at time t, in the straight configuration:

^tP : *axial conservative load*

L : *length of the rigid bar*

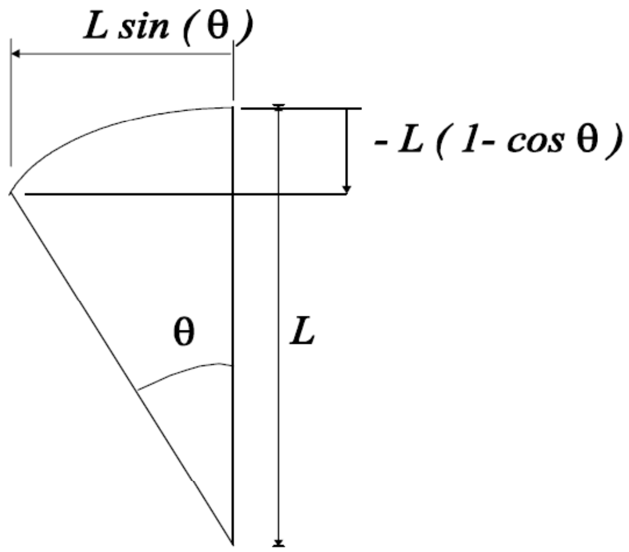
k : *stiffness of the linear spring*

k_T : *stiffness of the torsional spring*



The Principle of Virtual Work

Example 6.4 - continuation



$$For\ \theta << 1 \Rightarrow \Delta_P \approx -\frac{L\ \theta^2}{2}\ and\ L\ \sin(\theta) \approx L\ \theta$$

The potential of the external load is,

$${}^tG = -\ {}^tP\ \Delta_P = \frac{{}^tP\ L\ \theta^2}{2}\ .$$

The only deformable bodies are the springs; hence

$${}^t\mathrm{U} = \frac{1}{2}\ k\ (L\ \theta)^2\ +\ \frac{1}{2}\ k_T\ \theta^2$$

The Principle of Virtual Work

Example 6.4 - continuation

$$ {}^t_o\Pi = \frac{1}{2}\; k\; L^2\; \theta^2 + \frac{1}{2}\; k_T\; \theta^2 + \frac{{}^tP\; L\; \theta^2}{2} $$

and the equilibrium configuration is defined by

$$ \delta\, {}^t_\circ\Pi = 0 $$

which leads to,

$$ \left[k\; L^2\; \theta + \; k_T\; \theta + \; {}^tP\; L\; \theta\right]\; \delta\theta = 0\; . $$

Since $\delta\theta$ is arbitrary the bracket has to be zero.

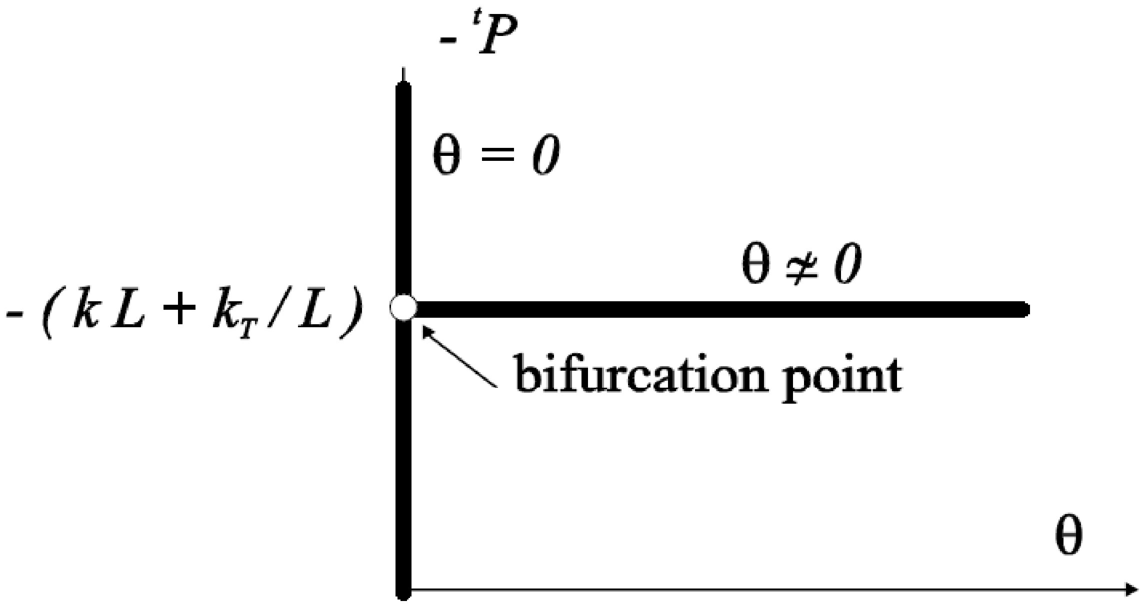
Two solutions are possible

(i) $\theta = 0$; that is to say, the straight (undeformed) configuration.
(ii) ${}^tP = -\left(k\; L + \frac{k_T}{L}\right)$.

The Principle of Virtual Work

Example 6.4 - continuation

For the second solution θ is undefined. We call this load value the critical value, P_{cr}, because at this load there are two branching solutions ($\theta = 0$ and $\theta \neq 0$).



Since in the above derivation the terms higher than θ^2 were neglected, we cannot assess anything about the branching equilibrium path.

The Principle of Virtual Work

Example 6.5. ◀◀◀◀◀

Postbuckling behavior.

We repeat the previous example derivation keeping terms higher than θ^2. By doing this, we get

$$\Delta_P = -\ L\ \ (1 - \cos\theta) \approx -\ L \left(\frac{\theta^2}{2} - \frac{\theta^4}{4!}\right)$$

$$^tG = -\ ^tP\ \Delta_P \approx\ ^tP\ L \left(\frac{\theta^2}{2} - \frac{\theta^4}{4!}\right)$$

$$^t\mathrm{U}_k = \frac{1}{2}\ k\ \ (L\ \ \sin\theta)^2 \approx \frac{1}{2}\ k\ L^2\ \left(\theta - \frac{\theta^3}{3!}\right)^2$$

$$^t\mathrm{U}_T = \frac{1}{2}\ k_T\ \theta^2\ .$$

Hence

$$^t_o\Pi = \frac{1}{2}\ k\ L^2\ \left(\theta^2 - \frac{\theta^4}{3} + \frac{\theta^6}{36}\right) + \frac{1}{2}\ k_T\ \theta^2 +\ ^tP\ L\ \frac{\theta^2}{2} -\ ^tP\ L\ \frac{\theta^4}{24}\ .$$

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Example 6.5 - continuation

$$ {}^{t}_{o}\Pi = \frac{1}{2}\, k\, L^2 \left(\theta^2 - \frac{\theta^4}{3} + \frac{\theta^6}{36}\right) + \frac{1}{2}\, k_T\, \theta^2 + {}^{t}P\, L\, \frac{\theta^2}{2} - {}^{t}P\, L\, \frac{\theta^4}{24} \ . $$

For the equilibrium configuration $\delta^{t}_{\circ}\Pi = 0$ and therefore,

$$ \left[k\, L^2 \left(\theta - \frac{2\,\theta^3}{3} + \frac{\theta^5}{12}\right) + k_T\, \theta + {}^{t}P\, L\, \theta - {}^{t}P\, L\, \frac{\theta^3}{6}\right] \delta\theta = 0 \ . $$

Since $\delta\theta$ is arbitrary, we get, neglecting terms higher than θ^3,

$$ \left[k\, L^2 \left(1 - \frac{2\,\theta^2}{3}\right) + k_T \ + {}^{t}P\, L \left(1 - \frac{\theta^2}{6}\right)\right] \theta = 0 $$

The Principle of Virtual Work

Example 6.5 - continuation

Two solutions are possible

(i) $\theta = 0$ the straight solution

(ii) $^{t}P = -\frac{k\ L\ \left(1-\frac{2\ \theta^2}{3}\right)+\frac{k_T}{L}}{1-\frac{\theta^2}{6}}$

In the second solution, for $\theta = 0,\ ^{t}P = P_{cr} = -\left(k\ L + \frac{k_T}{L}\right)$.

If we examine the case with $k_T = 0$, we get

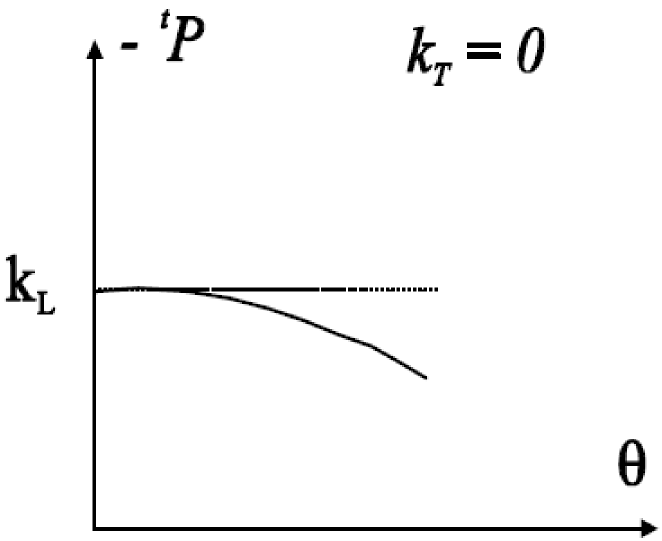
$$-\ ^{t}P = \frac{k\ L\ \left(1 - \frac{2\ \theta^2}{3}\right)}{1 - \frac{\theta^2}{6}}$$

The Principle of Virtual Work

Example 6.5 - continuation

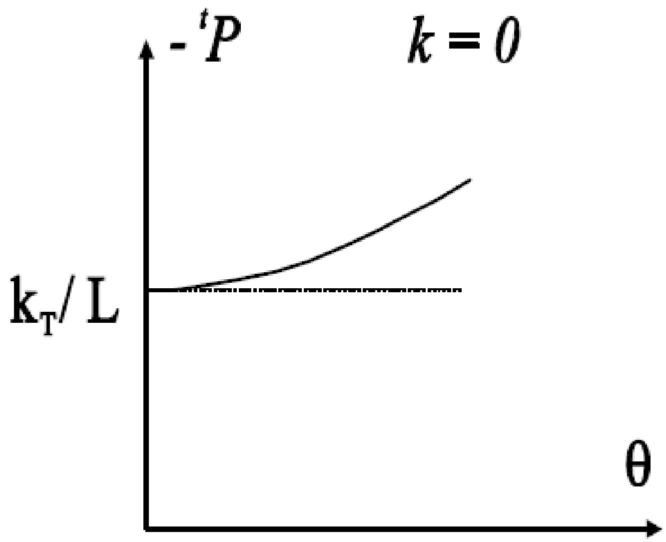
$k_T = 0$

$$-\,{}^{t}P = \frac{k\,L\left(1 - \frac{2\,\theta^2}{3}\right)}{1 - \frac{\theta^2}{6}}$$



$k_T = 0$

$$-\,{}^{t}P = \frac{k_T}{L\left(1 - \frac{\theta^2}{6}\right)}$$



Nonlinear Analysis Incremental Formulations

Incremental formulations

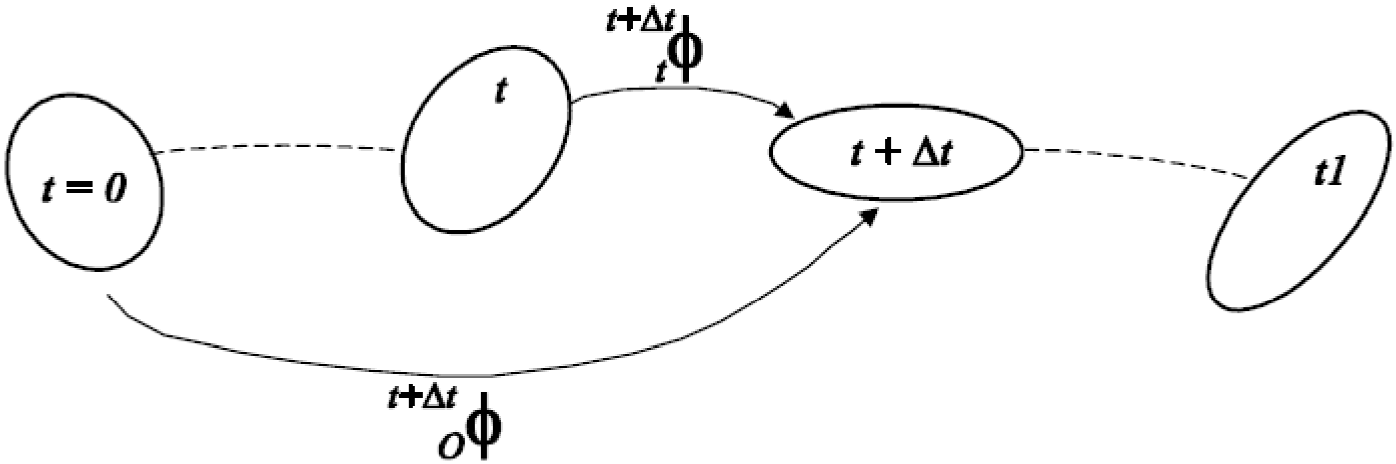


Fig. 6.2. Lagragian incremental analysis

Total Lagrangean Formulation

$$\int_{{}^{\circ}V} {}^{t+\Delta t}_{o}S_{IJ}\, \delta\, {}^{t+\Delta t}_{o}\varepsilon_{IJ}\, {}^{\circ}\mathrm{d}V = \delta^{t+\Delta t} W_{ext} \,.$$

$$ {}^{t+\Delta t}_{o}S_{IJ} = {}^{t}_{o}S_{IJ} + {}_{o}S_{IJ}$$

$$ {}^{t+\Delta t}_{o}\varepsilon_{IJ} = {}^{t}_{o}\varepsilon_{IJ} + {}_{o}\varepsilon_{IJ}$$

TLF

$$\int_{{}^{\circ}V} \left({}^{t}_{o}S_{IJ} + {}_{o}S_{IJ} \right) \delta_{o}\varepsilon_{IJ} \, {}^{\circ}\mathrm{d}V = \delta^{t+\Delta t} W_{ext}$$

Tangent constitutive relation:

$${}_{o}S_{IJ} = {}_{o}C_{IJKL} \, {}_{o}\varepsilon_{KL}$$

$$\int_{{}^{\circ}V} \left({}^{t}_{o}S_{IJ} + {}_{o}C_{IJKL} \, {}_{o}\varepsilon_{KL} \right) \delta_{o}\varepsilon_{IJ} \, {}^{\circ}\mathrm{d}V = \delta^{t+\Delta t} W_{ext}$$

TLF

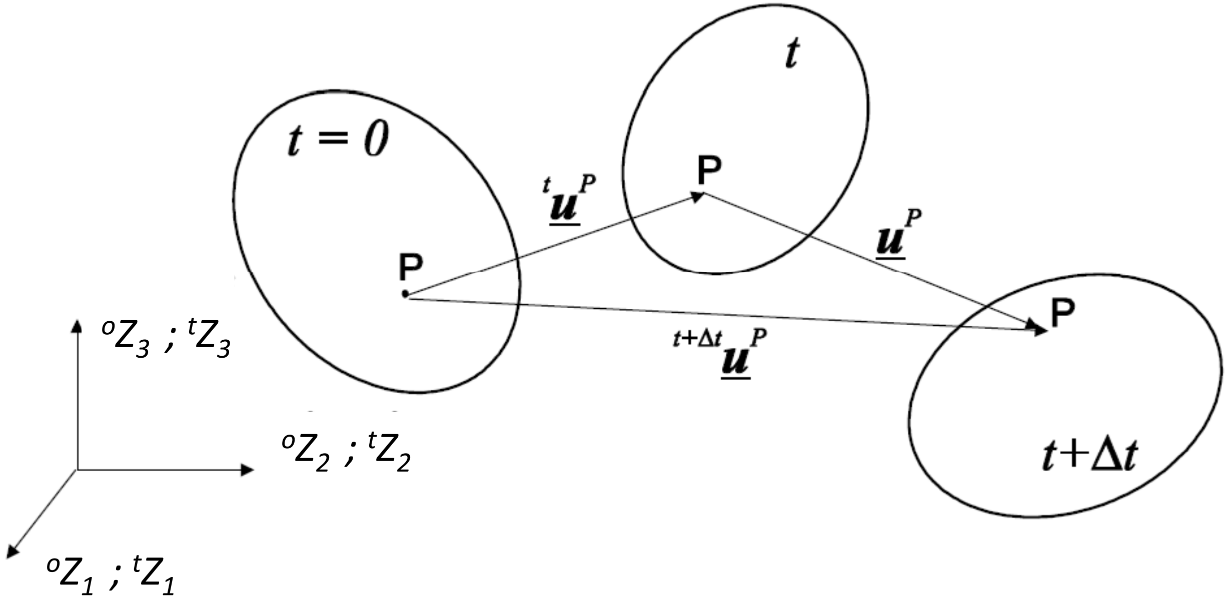


Fig. 6.3. Total Lagrangian formulation

$$^{t}\underline{\mathbf{u}}^{P} = {}^{t}\underline{\mathbf{x}}^{P} - {}^{\circ}\underline{\mathbf{x}}^{P}$$
$$\underline{\mathbf{u}}^{P} = {}^{t+\Delta t}\underline{\mathbf{u}}^{P} - {}^{t}\underline{\mathbf{u}}^{P}$$

TLF

$$ {}_{o}\varepsilon_{\alpha\beta} = \frac{1}{2}\left({}_{o}u_{\alpha,\beta} + {}_{o}u_{\beta,\alpha} + {}^{t}_{o}u_{\gamma,\alpha}\;{}_{o}u_{\gamma,\beta} + {}^{t}_{o}u_{\gamma,\beta}\;{}_{o}u_{\gamma,\alpha} + {}_{o}u_{\gamma,\alpha}\;{}_{o}u_{\gamma,\beta}\right) . \tag{6.28} $$

Total strain increment $\quad {}_{o}\varepsilon_{\alpha\beta} = {}_{o}e_{\alpha\beta} + {}_{o}\eta_{\alpha\beta}$

Linear increment $\quad {}_{o}e_{\alpha\beta} = \frac{1}{2}\left({}_{o}u_{\alpha,\beta} + {}_{o}u_{\beta,\alpha} + {}^{t}_{o}u_{\gamma,\alpha}\;{}_{o}u_{\gamma,\beta} + {}^{t}_{o}u_{\gamma,\beta}\;{}_{o}u_{\gamma,\alpha}\right)$

Nonlinear increment $\quad {}_{o}\eta_{\alpha\beta} = \frac{1}{2}\,{}_{o}u_{\gamma,\alpha}\;{}_{o}u_{\gamma,\beta}\,.$

TLF

$$\int_{{}^{\circ}V} \left[{}^{t}_{o}S_{\alpha\beta} + {}_{o}C_{\alpha\beta\gamma\delta}\left({}_{o}e_{\gamma\delta} + {}_{o}\eta_{\gamma\delta}\right)\right] \delta\left({}_{o}e_{\alpha\beta} + {}_{o}\eta_{\alpha\beta}\right) {}^{\circ}\mathrm{d}V = \delta^{t+\Delta t}W_{ext}$$

The above is the momentum balance equation at time $t + \Delta t$; which is a nonlinear equation in the incremental displacement vector. In order to solve it we use an iterative technique, in which the first step is the linearization of Eq. (6.30) (Bathe 1996). Keeping only up to the linear terms in $\underline{\mathbf{u}}$, we obtain the *linearized momentum balance equation*:

$$\int_{{}^{\circ}V} \left[{}_{o}C_{\alpha\beta\gamma\delta}\ {}_{o}e_{\gamma\delta}\ \delta_{o}e_{\alpha\beta} + {}^{t}_{o}S_{\alpha\beta}\ \delta_{o}\eta_{\alpha\beta}\right] {}^{\circ}\mathrm{d}V$$
$$= \delta^{t+\Delta t}W_{ext} - \int_{{}^{\circ}V} {}^{t}_{o}S_{\alpha\beta}\ \delta_{o}e_{\alpha\beta}\ {}^{\circ}\mathrm{d}V\ .$$

Updated Lagrangean Formulation

$$\int_{{}^tV} {}^{t+\Delta t}_{t}S_{IJ} \; \delta^{t+\Delta t}_{t}\varepsilon_{IJ} \; {}^{t}\mathrm{d}V = \delta^{t+\Delta t}W_{ext} \, .$$

$$ {}^{t+\Delta t}_{t}S_{IJ} = {}^{t}_{t}S_{IJ} + {}_{t}S_{IJ} $$

$$ {}^{t}_{t}S_{IJ} = {}^{t}\sigma_{IJ} $$

$$ {}^{t+\Delta t}_{t}\varepsilon_{IJ} = {}_{t}\varepsilon_{IJ} $$

ULF

$$\int_{^tV} \left({}^t\sigma_{IJ} + {}_tS_{IJ}\right) \delta\left({}_t\varepsilon_{IJ}\right) {}^t\mathrm{d}V = \delta^{t+\Delta t}W_{ext}$$

$${}_t\varepsilon_{\alpha\beta} = \frac{1}{2}\left({}_tu_{\alpha,\beta} + {}_tu_{\beta,\alpha} + {}_tu_{\gamma,\alpha}\ {}_tu_{\gamma,\beta}\right)$$

Total strain increment $\quad {}_t\varepsilon_{\alpha\beta} = {}_te_{\alpha\beta} + {}_t\eta_{\alpha\beta}$

Linear increment $\quad {}_te_{\alpha\beta} = \frac{1}{2}\left({}_tu_{\alpha,\beta} + {}_tu_{\beta,\alpha}\right)$

Nonlinear increment $\quad {}_t\eta_{\alpha\beta} = \frac{1}{2}\left({}_tu_{\gamma,\alpha}\ {}_tu_{\gamma,\beta}\right)$.

ULF

$$\int_{^tV} \left[{}^t\sigma_{\alpha\beta} + {}_tC_{\alpha\beta\gamma\delta} \left({}_te_{\gamma\delta} + {}_t\eta_{\gamma\delta}\right)\right] \delta\left({}_te_{\alpha\beta} + {}_t\eta_{\alpha\beta}\right) {}^t\mathrm{d}V = \delta^{t+\Delta t}W_{ext} . \tag{6.40}$$

The above is the momentum balance equation at time $t + \Delta t$; which is a nonlinear equation in the incremental displacement vector. Proceeding in the same way as in the total Lagrangian formulation we obtain the *linearized momentum balance equation* (Bathe 1996):

$$\int_{^tV} {}_tC_{\alpha\beta\gamma\delta}\ {}_te_{\gamma\delta}\ \delta_te_{\alpha\beta}\ {}^t\mathrm{d}V + \int_{^tV} {}^t\sigma_{\alpha\beta}\ \delta_t\eta_{\alpha\beta}\ {}^t\mathrm{d}V$$
$$= \delta^{t+\Delta t}W_{ext} - \int_{^tV} {}^t\sigma_{\alpha\beta}\ \delta_te_{\alpha\beta}\ {}^t\mathrm{d}V .$$

TLF and ULF

$$ {}_{o}^{t+\Delta t}S_{IJ} = \frac{{}^{o}\rho}{{}^{t+\Delta t}\rho}\; {}^{t+\Delta t}\sigma_{ij} \left({}_{o}^{t+\Delta t}X^{-1}\right)_{Ii} \left({}_{o}^{t+\Delta t}X^{-1}\right)_{Jj} $$

$$ {}_{t}^{t+\Delta t}S_{\hat{l}\hat{m}} = \frac{{}^{o}\rho}{{}^{t+\Delta t}\rho}\; {}^{t+\Delta t}\sigma_{ij} \left({}_{t}^{t+\Delta t}X^{-1}\right)_{\hat{l}i} \left({}_{t}^{t+\Delta t}X^{-1}\right)_{\hat{m}j} $$

$$ {}_{o}^{t+\Delta t}\underline{\underline{X}} = {}_{t}^{t+\Delta t}\underline{\underline{X}} \cdot {}_{o}^{t}\underline{\underline{X}} $$

$$ {}_{o}^{t+\Delta t}\underline{\underline{X}}^{-1} = {}_{o}^{t}\underline{\underline{X}}^{-1} \cdot {}_{t}^{t+\Delta t}\underline{\underline{X}}^{-1} $$

TLF and ULF

$$ {}^{t+\Delta t}_{o}S_{IJ} = \frac{{}^{o}\rho}{{}^{t+\Delta t}\rho}\; {}^{t+\Delta t}\sigma_{ij} \left({}^{t}_{o}X^{-1}\right)_{I\hat{l}} \left({}^{t+\Delta t}_{t}X^{-1}\right)_{\hat{l}i} \left({}^{t}_{o}X^{-1}\right)_{J\hat{m}} \left({}^{t}_{o}X^{-1}\right)_{\hat{m}J} $$

$$ {}^{t+\Delta t}_{o}S_{IJ} = \frac{{}^{o}\rho}{{}^{t+\Delta t}\rho}\frac{{}^{t+\Delta t}\rho}{{}^{t}\rho}\; {}^{t+\Delta t}_{o}S_{\hat{l}\hat{m}} \left({}^{t}_{o}X^{-1}\right)_{I\hat{l}} \left({}^{t}_{o}X^{-1}\right)_{J\hat{m}} $$

$$ {}^{t+\Delta t}_{o}S_{IJ} = \frac{{}^{o}\rho}{{}^{t}\rho}\; {}^{t+\Delta t}_{o}S_{\hat{l}\hat{m}} \left({}^{t}_{o}X^{-1}\right)_{I\hat{l}} \left({}^{t}_{o}X^{-1}\right)_{J\hat{m}} $$

TLF and ULF

For the increment

$$ {}^{t+\Delta t}_{o}S_{IJ} = \frac{{}^{o}\rho}{{}^{t}\rho}\; {}^{t+\Delta t}_{o}S_{\hat{l}\hat{m}} \left({}^{t}_{o}X^{-1}\right)_{I\hat{l}} \left({}^{t}_{o}X^{-1}\right)_{J\hat{m}} $$

TLF and ULF

$$ {}^{t+\Delta t}_{o}\underline{\varepsilon} = \frac{1}{2}\left[{}^{t+\Delta t}_{o}\underline{X}^{T} \cdot {}^{t+\Delta t}_{o}\underline{X} - {}^{o}\underline{g} \right] $$

$$ {}^{t+\Delta t}_{o}\underline{\underline{X}} = {}^{t+\Delta t}_{t}\underline{\underline{X}} \cdot {}^{t}_{o}\underline{\underline{X}} $$

$$ {}^{t+\Delta t}_{t}\underline{\underline{\varepsilon}} = \frac{1}{2}\left[{}^{t+\Delta t}_{t}\underline{\underline{X}}^{T} \cdot {}^{t+\Delta t}_{t}\underline{\underline{X}} - {}^{o}\underline{\underline{g}} \right] $$

$$ 2\ {}^{t+\Delta t}_{t}\underline{\varepsilon} + {}^{o}\underline{g} = {}^{t}_{o}\underline{X}^{T} \cdot {}^{t+\Delta t}_{t}\underline{X}^{T} \cdot {}^{t+\Delta t}_{t}\underline{X} \cdot {}^{t}_{o}\underline{X} $$

TLF and ULF

$$2\;{}^{t+\Delta t}_{o}\underline{\underline{\varepsilon}} + {}^{o}\underline{\underline{g}} = {}^{t}_{o}\underline{\underline{X}}^{T} \cdot \left(2\;{}^{t+\Delta t}_{t}\underline{\underline{\varepsilon}} + {}^{o}\underline{\underline{g}} \right) \cdot {}^{t}_{o}\underline{\underline{X}}$$

$$2\;{}^{t+\Delta t}_{o}\underline{\underline{\varepsilon}} - 2\;{}^{t}_{o}\underline{\underline{X}}^{T} \cdot {}^{t+\Delta t}_{t}\underline{\underline{\varepsilon}} \cdot {}^{t}_{o}\underline{\underline{X}} = {}^{t}_{o}\underline{\underline{X}}^{T} \cdot {}^{t}_{o}\underline{\underline{X}} - {}^{o}\underline{\underline{g}}$$

$${}^{t+\Delta t}_{o}\underline{\underline{\varepsilon}} - {}^{t}_{o}\underline{\underline{X}}^{T} \cdot \underbrace{{}^{t+\Delta t}_{t}\underline{\underline{\varepsilon}}}_{{}_{o}\underline{\underline{\varepsilon}}} \cdot {}^{t}_{o}\underline{\underline{X}} = {}^{t}_{o}\underline{\underline{\varepsilon}}$$

$${}_{o}\underline{\underline{\varepsilon}} = {}^{t}_{o}\underline{\underline{X}}^{T} \cdot {}_{t}\underline{\underline{\varepsilon}} \cdot {}^{t}_{o}\underline{\underline{X}}$$

TLF and ULF

It is easy to show that

$$ {}_{o}S_{IJ} = \frac{{}^{o}\rho}{{}^{t}\rho} \, {}_{t}S_{ij} \, \left({}^{t}_{o}X^{-1}\right)_{Ii} \, \left({}^{t}_{o}X^{-1}\right)_{Jj} $$

$$ {}_{o}\varepsilon_{IJ} = {}_{t}\varepsilon_{ij} \, {}^{t}_{o}X_{iI} \, {}^{t}_{o}X_{jJ} $$

and therefore if the same material is considered in both formulations the incremental constitutive tensors should be related,

$$ {}_{o}C_{IJKL} = \frac{{}^{o}\rho}{{}^{t}\rho} \, {}_{t}C_{mnpq} \, \left({}^{t}_{o}X^{-1}\right)_{Im} \left({}^{t}_{o}X^{-1}\right)_{Jn} \left({}^{t}_{o}X^{-1}\right)_{Kp} \left({}^{t}_{o}X^{-1}\right)_{Lq} $$