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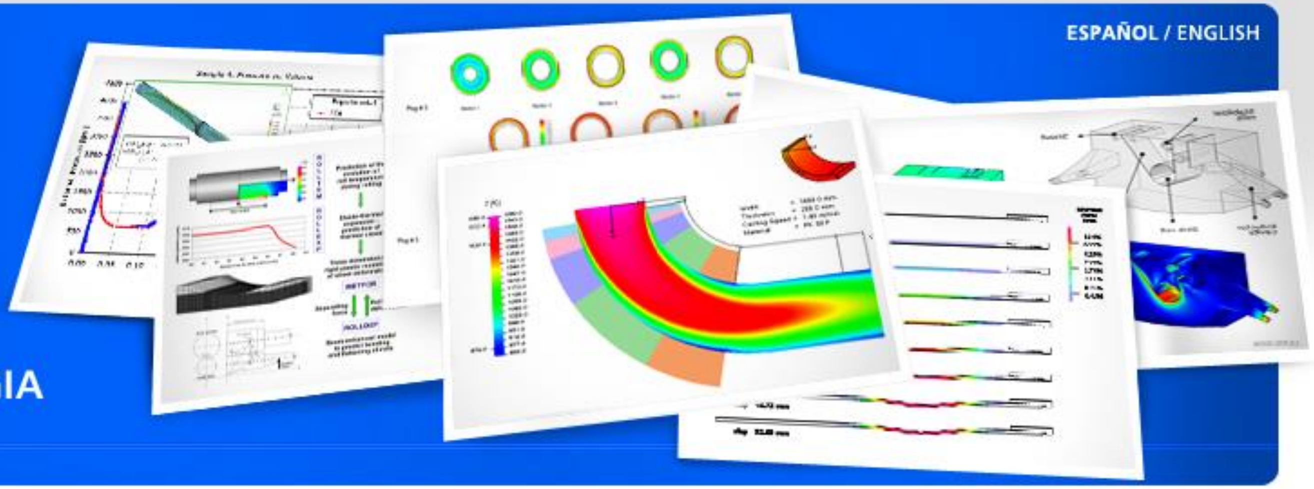

DE LA CIENCIA

A LA TECNOLOGIA

Advanced Topics in Computational Solid Mechanics. Industrial Applications

Section 8: General Nonlinear Shell Elements

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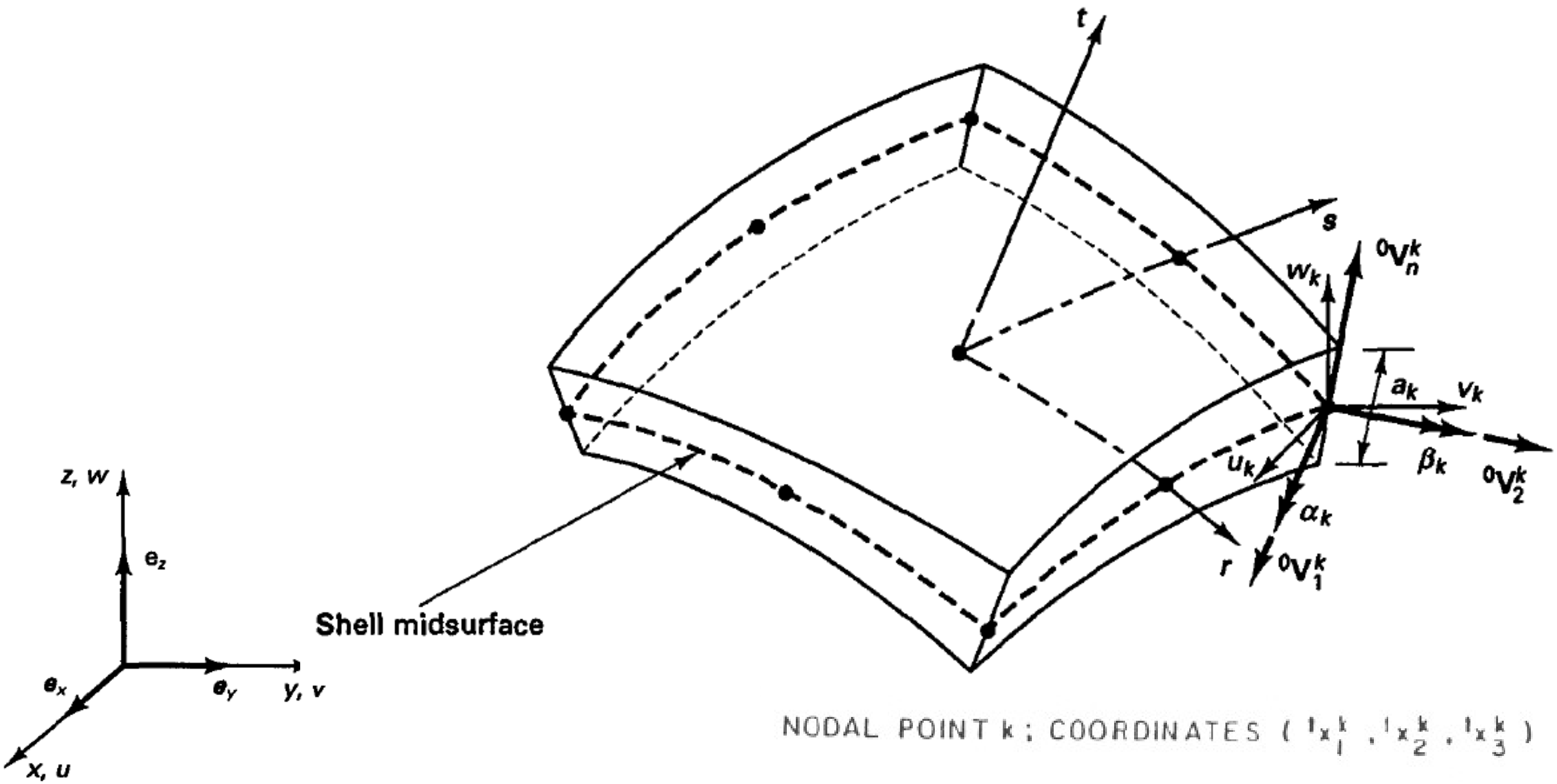
The Ahmad-Irons- Zienkiewicz Element C° Continuity



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The Ahmad-Irons- Zienkiewicz Element C° Continuity

$$ {}^{t}\underline{x} = h_k \; {}^{t}x_k + \frac{r_3}{2} h_k \; {}^{t}a_k \; {}^{t}\underline{V}_n{}^{k} $$

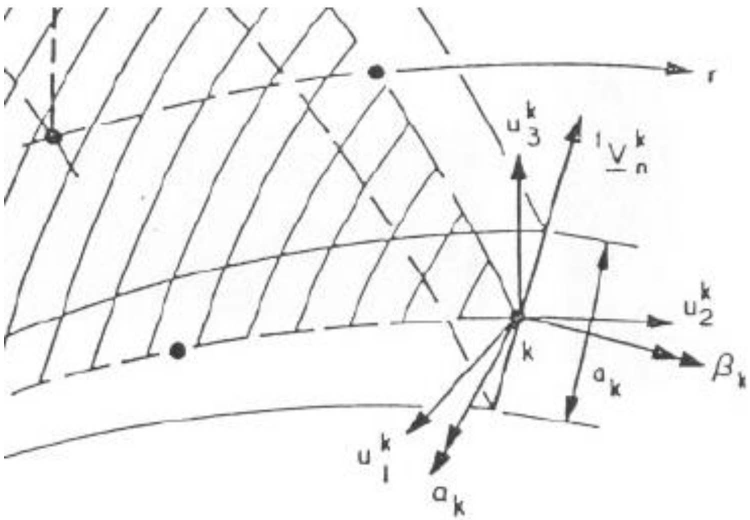
$$ \left| {}^{t}\underline{V}_n{}^{k} \right| = 1 $$

(t = 0 represents the reference undeformed configuration)

- The director vectors remain straight during the deformation process.
- The thickness remains constant during the deformation process $({}^{t+\Delta t}a_k = {}^{t}a_k = \cdots = {}^{\circ}a_k)$.

Incremental Displacements

$$\underline{\mathbf{u}} = {}^{t+\Delta t}\underline{\mathbf{x}} - {}^{t}\underline{\mathbf{x}} = h_k\, \underline{\mathbf{u}}_k + \frac{r_3}{2}\, h_k\, {}^t a_k\, (-\alpha_k\, {}^t\underline{\mathbf{V}}_2{}^k + \beta_k\, {}^t\underline{\mathbf{V}}_1{}^k)$$



- If $\quad {}^t\underline{\mathbf{e}}_2 \times {}^t\underline{\mathbf{V}}_n{}^k \neq \underline{\mathbf{0}}$

$${}^t\underline{\mathbf{V}}_1{}^k = \frac{{}^t\underline{\mathbf{e}}_2 \times {}^t\underline{\mathbf{V}}_n{}^k}{\left|{}^t\underline{\mathbf{e}}_2 \times {}^t\underline{\mathbf{V}}_n{}^k\right|}$$

$${}^t\underline{\mathbf{V}}_2{}^k = {}^t\underline{\mathbf{V}}_n{}^k \times {}^t\underline{\mathbf{V}}_1{}^k$$

- If $\quad {}^t\underline{\mathbf{e}}_2 \times {}^t\underline{\mathbf{V}}_n{}^k = \underline{\mathbf{0}}$

$${}^t\underline{\mathbf{V}}_1{}^k = {}^t\underline{\mathbf{e}}_3$$

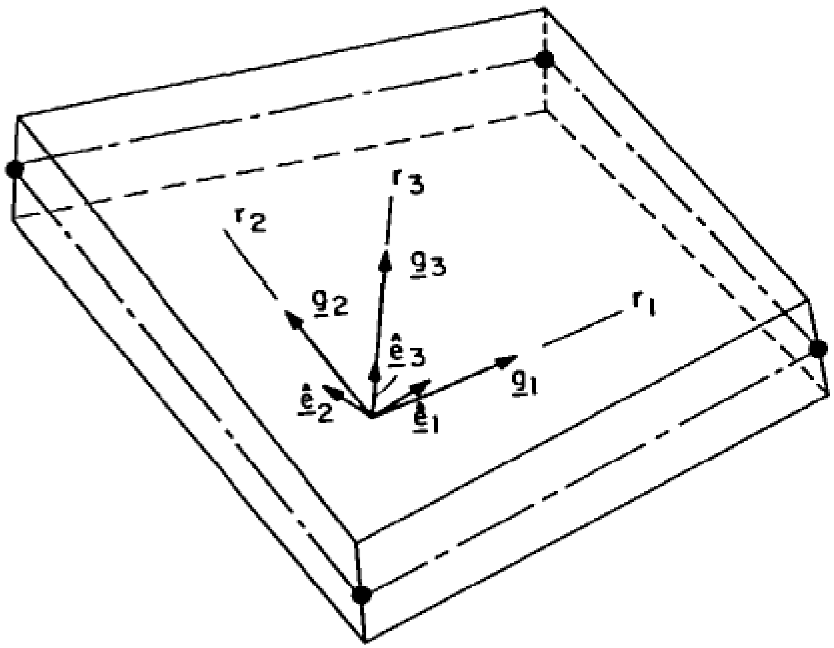
$${}^t\underline{\mathbf{V}}_2{}^k = {}^t\underline{\mathbf{e}}_1$$

Incremental Displacements

A special definition of ${}^{t}\underline{\mathbf{V}}_2{}^{k}$ and ${}^{t}\underline{\mathbf{V}}_1{}^{k}$ is used for the case when ${}^{t}\underline{\mathbf{V}}_n{}^{k}$ is parallel to the y-axis.

5 d.o.f. / node

Constitutive Relations



$\hat{e}_3 = \frac{g_3}{|g_3|}$; $\hat{e}_1 = \frac{g_2 \times \hat{e}_3}{|g_2 \times \hat{e}_3|}$; $\hat{e}_2 = \hat{e}_3 \times \hat{e}_1$

Figure 3 Local Cartesian coordinate system used

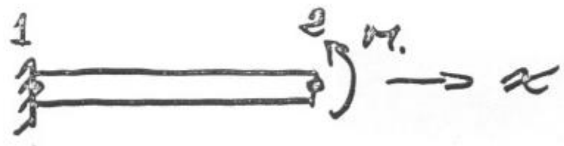
$$\hat{\sigma}_{33} = 0$$

$$\hat{\epsilon}_{33} = 0$$

$$\tilde{C}^{ijkl} = (\mathbf{g}^i \cdot \hat{\mathbf{e}}_m)(\mathbf{g}^j \cdot \hat{\mathbf{e}}_n)(\mathbf{g}^k \cdot \hat{\mathbf{e}}_o)(\mathbf{g}^l \cdot \hat{\mathbf{e}}_p)\hat{C}^{mnop}$$

$$\tilde{\tau}^{ij} = \tilde{C}^{ijkl}\tilde{\varepsilon}_{kl}$$

The Locking Problem



$$w = h_1 w_1 + h_2 w_2 \quad (linear)$$

$$\theta = h_1 \theta_1 + h_2 \theta_2 \quad (linear)$$

$$\underbrace{\frac{dw}{dx}}_{const} - \theta = 0$$

$$\Downarrow$$

$$\theta = const = 0 \; (B.C.)$$

LOCKING

The Locking Problem

$$\boxed{\pi = \frac{h^3}{2}\int_A {}^t\underline{x}^T \, \underline{C}_b \, {}^t\underline{x} \; dA + \alpha \int_A {}^t\underline{\gamma}^T \, \underline{C}_s \, {}^t\underline{\gamma} \; dA}$$

$${}^t\underline{x} = \begin{bmatrix} \dfrac{\partial \alpha}{\partial x} \\ -\dfrac{\partial \beta}{\partial y} \\ \dfrac{\partial \alpha}{\partial y} - \dfrac{\partial \beta}{\partial x} \end{bmatrix}$$

$${}^t\underline{\gamma} = \frac{1}{L}\begin{bmatrix} \dfrac{\partial w}{\partial y} - \beta \\ \dfrac{\partial w}{\partial x} + \alpha \end{bmatrix}$$

$$\underline{C}_b = \frac{E}{12\left(1-\nu^2\right)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}$$

$$\underline{C}_s = \frac{k\,E}{2\left(1+\nu\right)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

E : Young's modulus

ν : Poisson ratio

k : shear correction factor

$\alpha = \frac{L}{h} \xrightarrow[h \to 0]{} \infty$

where w = transverse displacement of the plate

h = plate thickness

The Locking Problem

In curved elements a similar problem appears if the interpolation functions cannot represent states of zero membrane deformation (membrane locking)

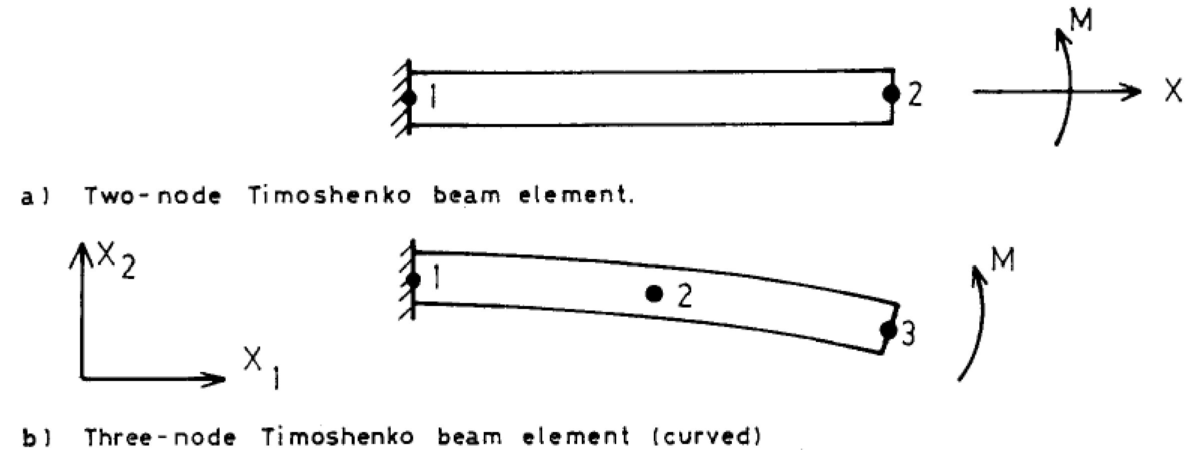
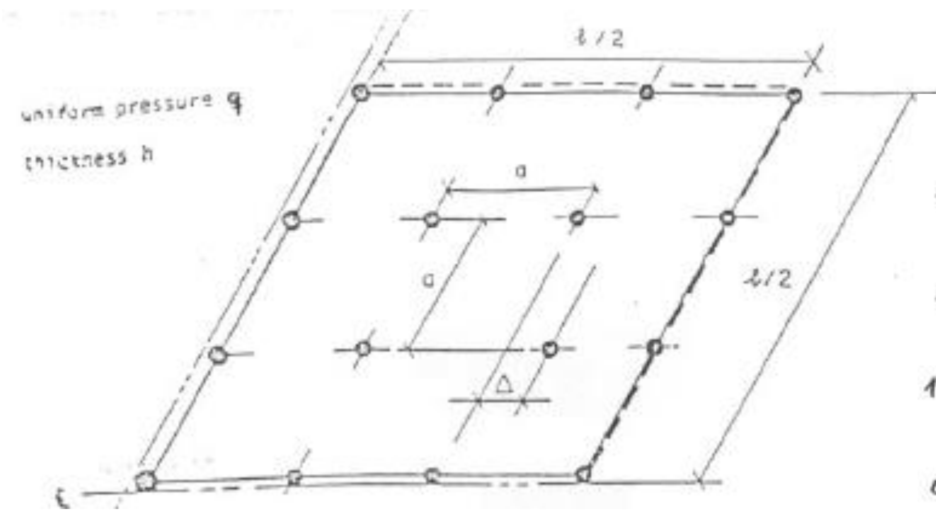


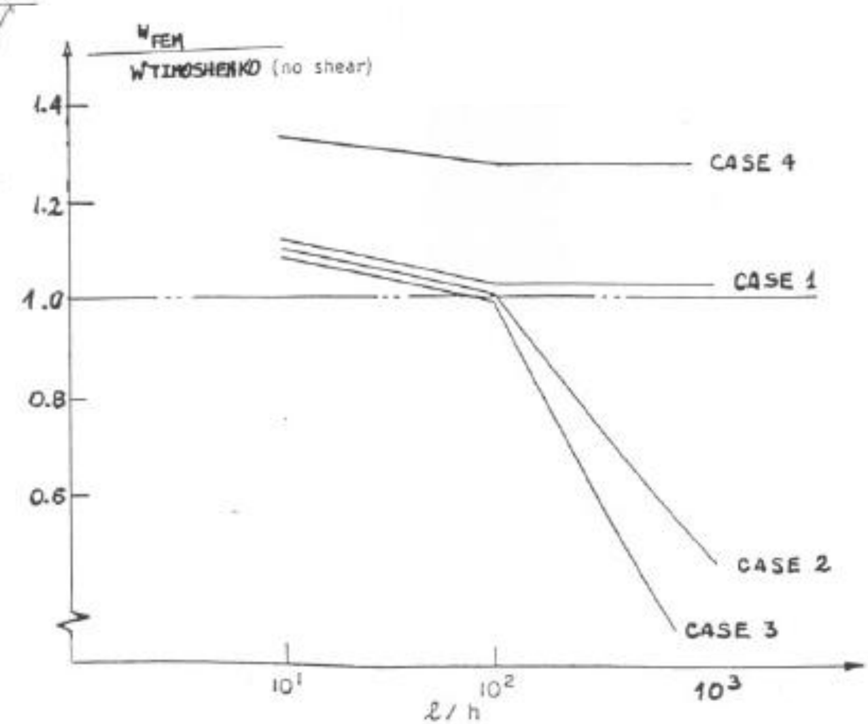
Figure 2. Simple problems illustrating the locking phenomenon

$$\pi = \frac{E\,I}{2}\left[\int_0^L \theta_{,s}{}^2\,ds + \xi_M \int_0^L u_{s,s}{}^2\,ds + \xi_S \int_0^L (u_{n,s} - \theta)^2\,ds\right] - V$$

The Locking Problem

Simply-supported plate

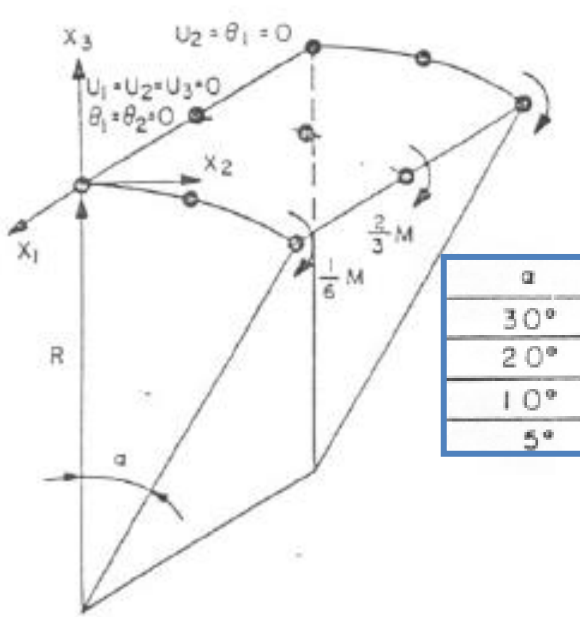




Case	Δ/a	Integration
1	0	4 x 4 x 2
2	1/50	4 x 4 x 2
3	1/20	4 x 4 x 2
4	1/20	3 x 3 x 2

The Locking Problem

Curved cantilever

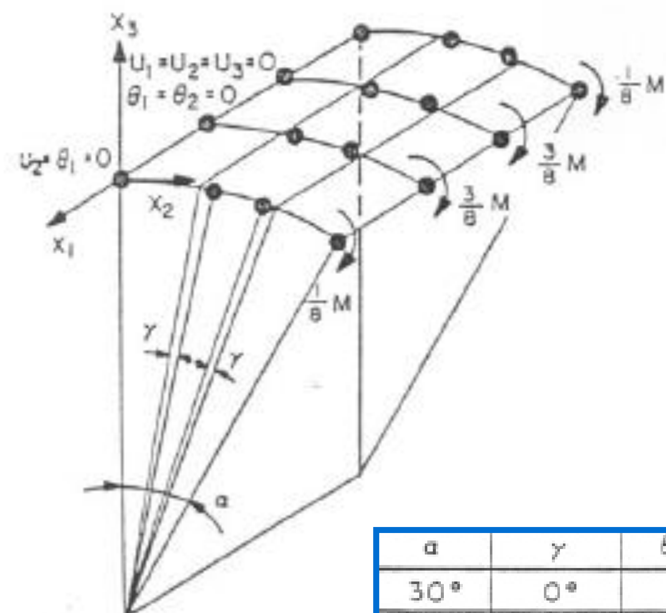


R = 20
h (thickness) = 0.2
E = 2.1 x 10^6
ν = 0.3
Int = 4x4x2

α	θ_{FE}/θ_{TH}
30°	0.02
20°	0.09
10°	0.62
5°	0.96

3 x 10° elements $\theta_{FE}/\theta_{TH} = 0.61$

6 x 5° elements $\theta_{FE}/\theta_{TH} = 0.96$



α	γ	θ_{FE}/θ_{TH}
30°	0°	0.91
30°	4°	0.01

The Figures are not to scale

The MITC4 Element

The MITC4 (mixed interpolation of tensorial components, four nodes) is a general shell element with the following features:

- It can be used in non-flat geometries (it is a shell element rather than a plate element)
- It can be used in general nonlinear analyses (material nonlinear analyses and geometrical nonlinear analyses but small strains)
- It does not lock and it does not present spurious rigid body modes.
- It can be used for thin and moderately thick shells.

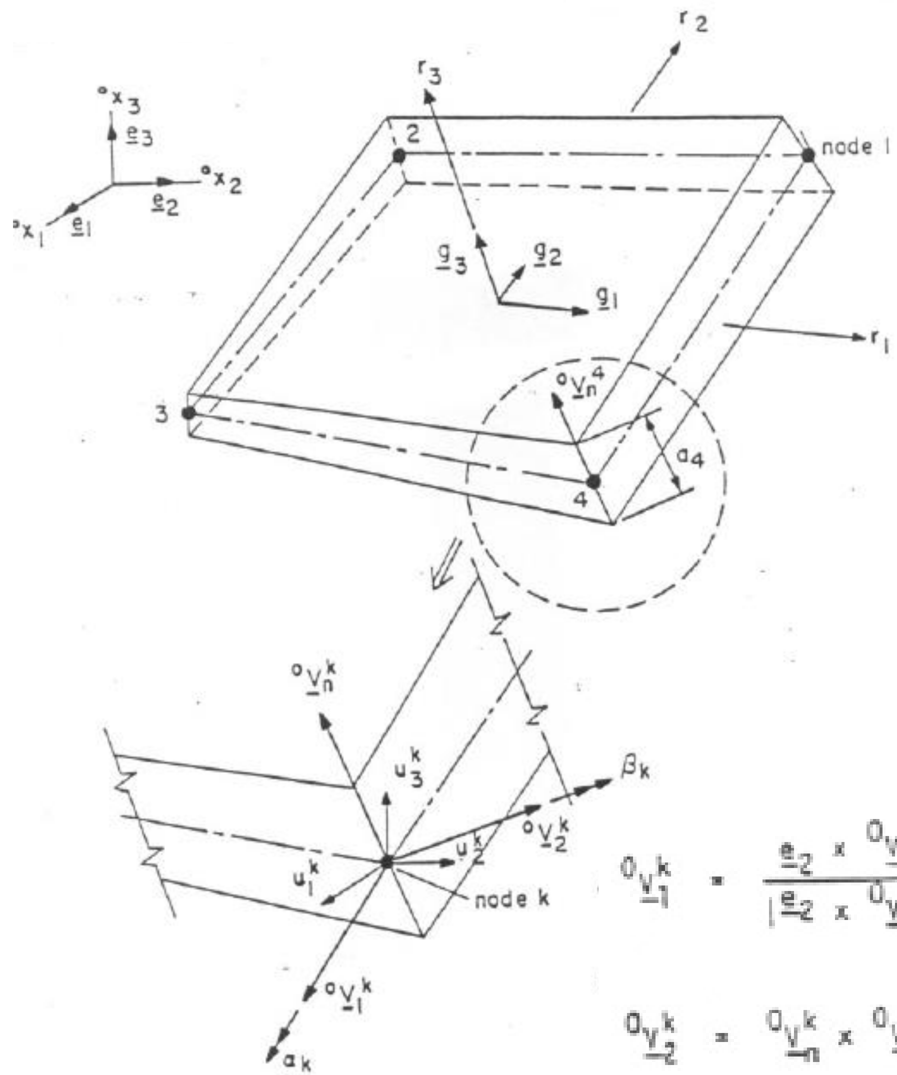
The MITC4 Element

Four node shell element

$$\underline{g}_i = \frac{\partial \underline{x}}{\partial r_i}$$

$${}^l x_i = h_k \; {}^l x_i^{\,k} + \frac{r_3}{2} h_k \, a_k \; {}^l V_{ni}^{\,k}$$

$l = 0$ undeformed configuration
$l = t$ deformed configuration



$${}^0\underline{V}_1^k = \frac{\underline{e}_2 \times {}^0\underline{V}_n^k}{|\underline{e}_2 \times {}^0\underline{V}_n^k|}$$

$${}^0\underline{V}_2^k = {}^0\underline{V}_n^k \times {}^0\underline{V}_1^k$$

The MITC4 Element

$$u_i = h_k\, u_i^{\,k} + \frac{r_3}{2} h_k\, a_k \left(-\alpha_k\, {}^{l}V_{2i}^{\;k} + \beta_k\, {}^{l}V_{1i}^{\;k} \right)$$

In the natural coordinate system, the strain tensor is:

$$\boxed{\underline{\underline{\varepsilon}} = \tilde{\varepsilon}_{ij}\, \underline{g}^{\,i}\, \underline{g}^{\,j}}$$

where $\tilde{\varepsilon}_{ij}$ are the strain tensor components and $\underline{g}^{\,i}$ the contravariant base vectors

The MITC4 Element

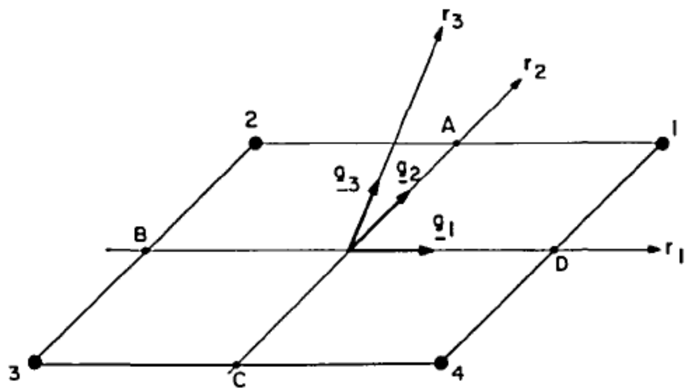
- The usual A-I-Z interpolations for the displacements/rotations, Eqn. (3)

- The “in-layer” strains ($\tilde{\varepsilon}_{11}$, $\tilde{\varepsilon}_{22}$ and $\tilde{\varepsilon}_{33}$) are directly calculated from the displacement interpolation using the kinematic relations.

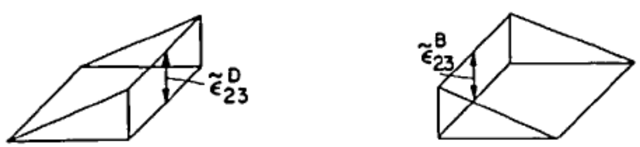
- The transverse shear strains $\tilde{\varepsilon}_{13}$ and $\tilde{\varepsilon}_{33}$ are interpolated using the interpolations shown in Fig. 2

The MITC4 Element

Interpolation functions for the transverse shear strains



$\tilde{\varepsilon}_{13}$ interpolation



$\tilde{\varepsilon}_{23}$ interpolation

$$\tilde{\varepsilon}_{13} = \frac{1}{2}(1+r_2)\,\tilde{\varepsilon}_{13}\Big|_A^{DI} + \frac{1}{2}(1-r_2)\,\tilde{\varepsilon}_{13}\Big|_C^{DI}$$

$$\tilde{\varepsilon}_{23} = \frac{1}{2}(1+r_1)\,\tilde{\varepsilon}_{23}\Big|_D^{DI} + \frac{1}{2}(1-r_1)\,\tilde{\varepsilon}_{23}\Big|_B^{DI}$$

MITC4 - TLF

$$\int_{{}^0V} {}^{t+\Delta t}_{\ 0}\tilde{S}^{ij}\,\delta\,{}^{t+\Delta t}_{\ 0}\tilde{\varepsilon}_{ij}\,{}^0dV = {}^{t+\Delta t}\mathscr{R}$$

$${}^{t+\Delta t}_{\ 0}\tilde{S}^{ij} = {}^{t}_{0}\tilde{S}^{ij} + {}_0\tilde{S}^{ij}$$

$${}^{t+\Delta t}_{\ 0}\tilde{\varepsilon}_{ij} = {}^{t}_{0}\tilde{\varepsilon}_{ij} + {}_0\tilde{\varepsilon}_{ij}$$

$$\int_{{}^0V} {}_0\tilde{C}^{ijkl}\,{}_0\tilde{e}_{kl}\,\delta\,{}_0\tilde{e}_{ij}\,{}^0dV + \int_{{}^0V} {}^{t}_{0}\tilde{S}^{ij}\,\delta\,{}_0\tilde{\eta}_{ij}\,{}^0dV$$

$$= {}^{t+\Delta t}\mathscr{R} - \int_{{}^0V} {}^{t}_{0}\tilde{S}^{ij}\,\delta\,{}_0\tilde{e}_{ij}\,{}^0dV$$

MITC4 - TLF

$$\boxed{{}_0\tilde{e}_{ii} = h_{k,i}\,{}^t\mathbf{g}_i \cdot \mathbf{u}_k + \frac{r_3}{2} a_k h_{k,i}(-\alpha_k\,{}^t\mathbf{g}_i \cdot {}^t\mathbf{V}_2^k + \beta_k\,{}^t\mathbf{g}_i \cdot {}^t\mathbf{V}_1^k)}$$

$$(i = 1,2)$$

$$\boxed{\begin{aligned}{}_0\tilde{\eta}_{ii} = {} & \tfrac{1}{2} h_{k,i} h_{p,i} \mathbf{u}_k \cdot \mathbf{u}_p + \frac{r_3}{2} h_{k,i} h_{p,i} a_p(-\alpha_p\,{}^t\mathbf{V}_2^p \cdot \mathbf{u}_k + \beta_p\,{}^t\mathbf{V}_1^p \cdot \mathbf{u}_k) + \\ & \frac{(r_3)^2}{8} h_{k,i} h_{p,i} a_k a_p(-\alpha_k\,{}^t\mathbf{V}_2^k + \beta_k\,{}^t\mathbf{V}_1^k) \cdot (-\alpha_p\,{}^t\mathbf{V}_2^p + \beta^{p}\,{}^t\mathbf{V}_1^p)\end{aligned}}$$

$$ {}_0\tilde{e}_{12} = \tfrac{1}{2}[h_{k,2}\,{}^t\mathbf{g}_1 \cdot \mathbf{u}_k + h_{k,1}\,{}^t\mathbf{g}_2 \cdot \mathbf{u}_k + \frac{r_3}{2} h_{k,2} a_k(-\alpha_k\,{}^t\mathbf{V}_2^k \cdot {}^t\mathbf{g}_1 + \beta_k\,{}^t\mathbf{V}_1^k \cdot {}^t\mathbf{g}_1) + \frac{r_3}{2} h_{k,1} a_k(-\alpha_k\,{}^t\mathbf{V}_2^k \cdot {}^t\mathbf{g}_2 + \beta_k\,{}^t\mathbf{V}_1^k \cdot {}^t\mathbf{g}_2)]$$

$$\begin{aligned}{}_0\tilde{\eta}_{12} = {} & \tfrac{1}{2}[h_{k,1} h_{p,2} \mathbf{u}_k \cdot \mathbf{u}_p + \frac{r_3}{2} h_{k,1} h_{p,2} a_p(-\alpha_p\,{}^t\mathbf{V}_2^p \cdot \mathbf{u}_k + \beta_p\,{}^t\mathbf{V}_1^p \cdot \mathbf{u}_k) + \\ & \frac{r_3}{2} h_{k,1} h_{p,2} a_k(-\alpha_k\,{}^t\mathbf{V}_2^k \cdot \mathbf{u}_p + \beta_k\,{}^t\mathbf{V}_1^k \cdot \mathbf{u}_p) + \frac{(r_3)^2}{4} h_{k,1} h_{p,2} a_k a_p(-\alpha_k\,{}^t\mathbf{V}_2^k + \beta_k\,{}^t\mathbf{V}_1^k) \cdot (-\alpha_p\,{}^t\mathbf{V}_2^p + \beta_p\,{}^t\mathbf{V}_1^p)]\end{aligned}$$

MITC4 - TLF

$$
{}_0\tilde{e}_{13} = \tfrac{1}{8}(1+r_2)\left[{}^t g^{A}_{3i}(u^1_i - u^2_i) + \tfrac{1}{2}{}^t g^{A}_{1i}(-\alpha_1 a_1\, {}^tV^1_{2i} + \beta_1 a_1\, {}^tV^1_{1i} - \alpha_2 a_2\, {}^tV^2_{2i} + \beta_2 a_2\, {}^tV^2_{1i})\right] +
$$

$$
\tfrac{1}{8}(1-r_2)\left[{}^t g^{C}_{3i}(u^4_i - u^3_i) + \tfrac{1}{2}{}^t g^{C}_{1i}(-\alpha_4 a_4\, {}^tV^4_{2i} + \beta_4 a_4\, {}^tV^4_{1i} - \alpha_3 a_3\, {}^tV^3_{2i} + \beta_3 a_3\, {}^tV^3_{1i})\right]
$$

$$
{}_0\tilde{\eta}_{13} = \tfrac{1}{32}(1+r_2)\left[(-\alpha_1 a_1\, {}^tV^1_{2i} + \beta_1 a_1\, {}^tV^1_{1i} - \alpha_2 a_2\, {}^tV^2_{2i} + \beta_2 a_2\, {}^tV^2_{1i})(u^1_i - u^2_i)\right] +
$$

$$
\tfrac{1}{32}(1-r_2)\left[(-\alpha_4 a_4\, {}^tV^4_{2i} + \beta_4 a_4\, {}^tV^4_{1i} - \alpha_3 a_3\, {}^tV^3_{2i} + \beta_3 a_3\, {}^tV^3_{1i})(u^4_i - u^3_i)\right]
$$

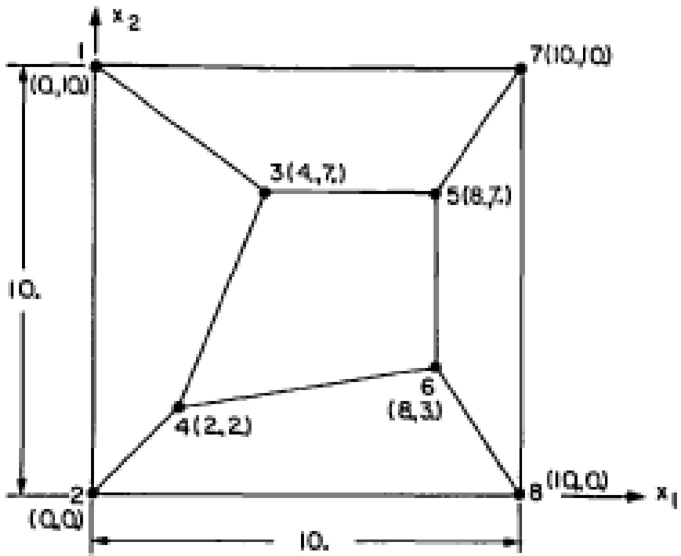
$$
{}_0\tilde{e}_{23} = \tfrac{1}{8}(1+r_1)\left[{}^t g^{D}_{3i}(u^1_i - u^4_i) + \tfrac{1}{2}{}^t g^{D}_{2i}(-\alpha_1 a_1\, {}^tV^1_{2i} + \beta_1 a_1\, {}^tV^1_{1i} - \alpha_4 a_4\, {}^tV^4_{2i} + \beta_4 a_4\, {}^tV^4_{1i})\right] +
$$

$$
\tfrac{1}{8}(1-r_1)\left[{}^t g^{B}_{3i}(u^2_i - u^3_i) + \tfrac{1}{2}{}^t g^{B}_{2i}(-\alpha_2 a_2\, {}^tV^2_{2i} + \beta_2 a_2\, {}^tV^2_{1i} - \alpha_3 a_3\, {}^tV^3_{2i} + \beta_3 a_3\, {}^tV^3_{1i})\right]
$$

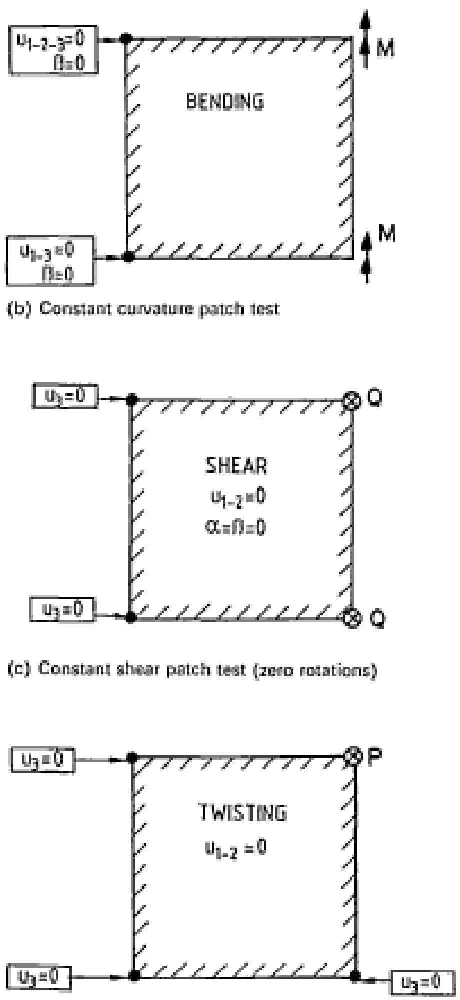
$$
{}_0\tilde{\eta}_{23} = \tfrac{1}{32}(1+r_1)\left[(-\alpha_1 a_1\, {}^tV^1_{2i} + \beta_1 a_1\, {}^tV^1_{1i} - \alpha_4 a_4\, {}^tV^4_{2i} + \beta_4 a_4\, {}^tV^4_{1i})(u^1_i - u^4_i)\right] +
$$

$$
\tfrac{1}{32}(1-r_1)\left[(-\alpha_2 a_2\, {}^tV^2_{2i} + \beta_2 a_2\, {}^tV^2_{1i} - \alpha_3 a_3\, {}^tV^3_{2i} + \beta_3 a_3\, {}^tV^3_{1i})(u^2_i - u^3_i)\right]
$$

MITC4 – The Patch Test



(a) Patch test mesh



(b) Constant curvature patch test

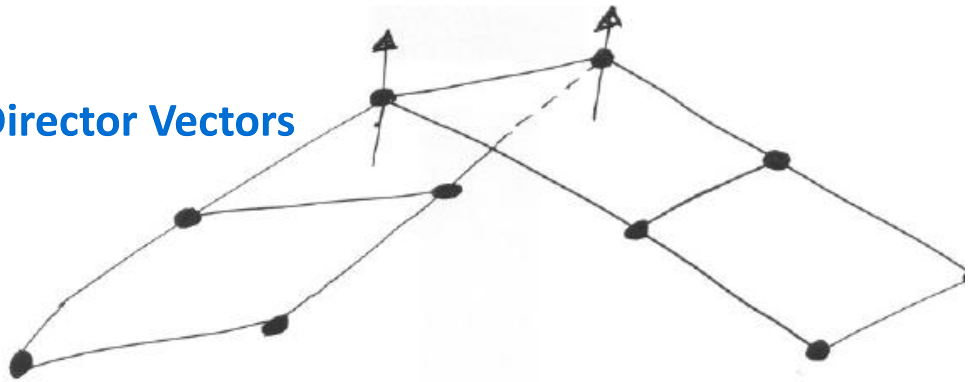
(c) Constant shear patch test (zero rotations)

(d) Constant twist patch test

MITC4 – Modeling Details

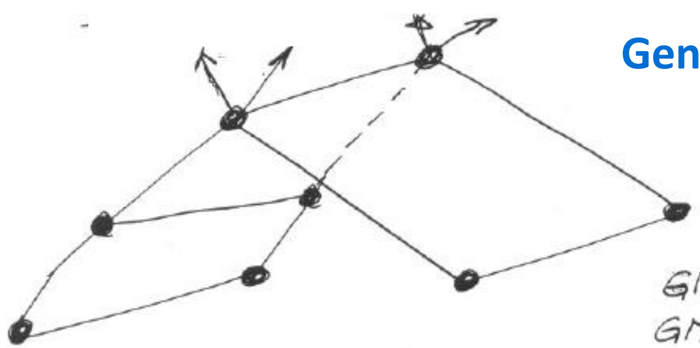
Shell Intersections

Director Vectors



DV6 → NO!!
DV5 → O.K.

Generated Normals

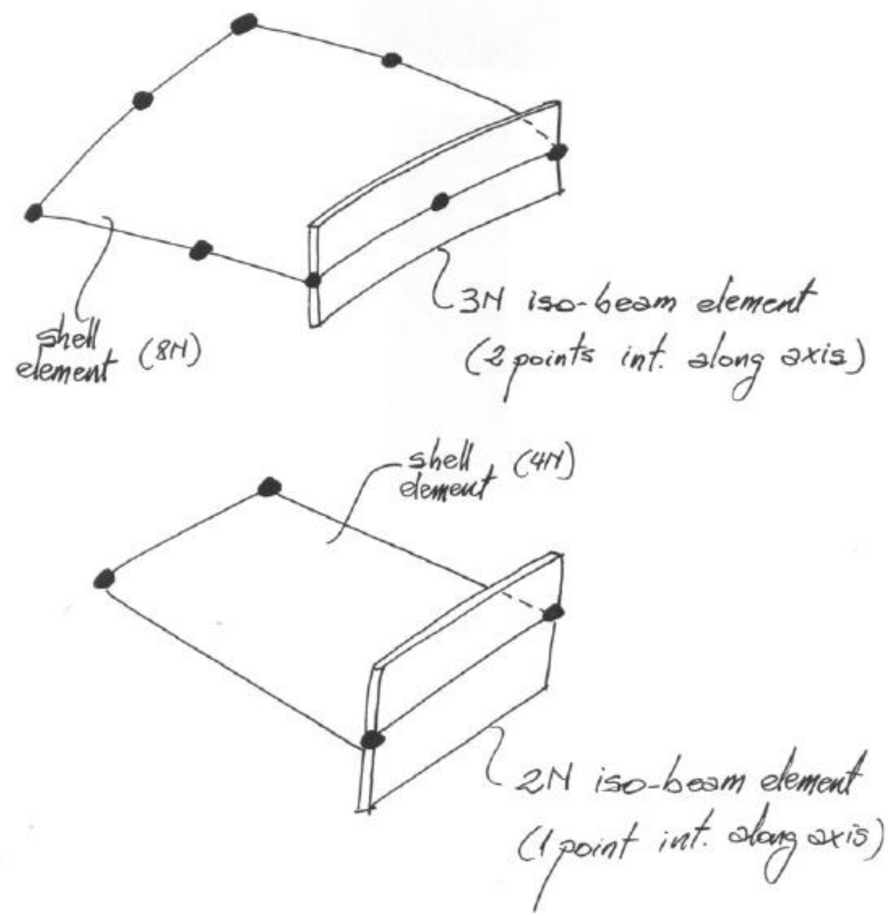


GN6 → O.K.
GN5 → NO!!

![SIM&TEC logo]

MITC4 – Modeling Details

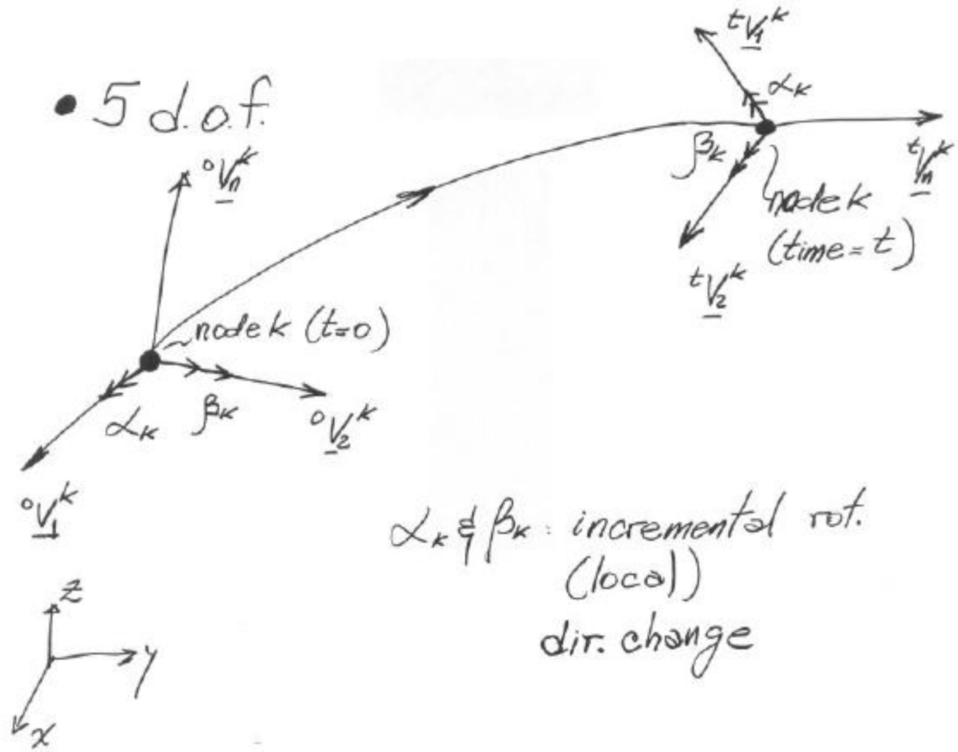
Modelling of stiffened shells



MITC4 – Modeling Details

Rotational boundary conditions in nonlinear analyses

- 5 d.o.f.



α_k & β_k : incremental rot. (local)
dir. change

- 6 d.o.f.
global always

MITC4 – Finite Rotations

$$ {}^{t+\Delta t}\underline{\mathbf{V}}_n^{\,k} = {}^{t+\Delta t}_{\ t}\underline{\underline{\mathbf{R}}}^k \cdot {}^{t}\underline{\mathbf{V}}_n^{\,k} $$

$$ \left[{}^{t+\Delta t}_{\ t}R^k\right] = [I_3] + \frac{\sin\left(\theta^k\right)}{\theta^k}\left[\Theta^k\right] + \frac{1}{2}\left[\frac{\sin\left(\theta^k\backslash 2\right)}{\left(\theta^k\backslash 2\right)}\right]^2\left[\Theta^k\right]^2 $$

In the above $\theta^k = \left[\left({\theta_1}^k\right)^2 + \left({\theta_2}^k\right)^2\right]^{\frac{1}{2}}$

$$ \left[\Theta^k\right] = \begin{bmatrix} 0 & 0 & {\theta_2}^k \\ 0 & 0 & -{\theta_1}^k \\ -{\theta_2}^k & {\theta_1}^k & 0 \end{bmatrix} $$

- For infinitesimal incremental rotations ${\theta_1}^k$ and ${\theta_2}^k$ are independent infinitesimal rotations around ${}^{t}\underline{\mathbf{V}}_1^{\,k}$ and ${}^{t}\underline{\mathbf{V}}_2^{\,k}$ respectively.
- For finite incremental rotations ${\theta_1}^k$ and ${\theta_2}^k$ *are not independent rotations*, they are the two variables that define the rotation tensor.
- As in the infinitesimal rotations case we only have 5 d.o.f. / node.

Finite Rotations - Linearization

$$\left[{}_{t}^{t+\Delta t}R^{k}\right] = [I_3] + \left[\Theta^k\right] + \frac{1}{2!}\left[\Theta^k\right]^2 + \cdots \tag{33}$$

Using the above in Eqn. (31) we get

$${}^{t+\Delta t}\underline{\mathbf{V}}_n{}^k - {}^{t}\underline{\mathbf{V}}_n{}^k = \underline{\theta}^k \times {}^{t}\underline{\mathbf{V}}_n{}^k + \frac{1}{2}\underline{\theta}^k \times \left(\underline{\theta}^k \times {}^{t}\underline{\mathbf{V}}_n{}^k\right) + h.o.t. \tag{34.a}$$

where we defined

$$\underline{\theta}^k = \theta_1{}^k\,{}^{t}\underline{\mathbf{V}}_1{}^k + \theta_2{}^k\,{}^{t}\underline{\mathbf{V}}_2{}^k \tag{34.b}$$

$${}^{t+\Delta t}\underline{\mathbf{V}}_n{}^k - {}^{t}\underline{\mathbf{V}}_n{}^k = \left(\theta_2{}^k\,{}^{t}\underline{\mathbf{V}}_1{}^k - \theta_1{}^k\,{}^{t}\underline{\mathbf{V}}_2{}^k\right) - \frac{1}{2}\left[(\theta_1{}^k)^2 + (\theta_2{}^k)^2\right]{}^{t}\underline{\mathbf{V}}_n{}^k + h.o.t.$$

$$\underline{\mathbf{u}} = h_k\,\underline{\mathbf{u}}_k + \frac{r_3}{2}h_k\,{}^{t}a_k\left(-\theta_1{}^k\,{}^{t}\underline{\mathbf{V}}_2{}^k + \theta_2{}^k\,{}^{t}\underline{\mathbf{V}}_1{}^k\right) - \frac{r_3}{4}h_k\,{}^{t}a_k\left[(\theta_1{}^k)^2 + (\theta_2{}^k)^2\right]{}^{t}\underline{\mathbf{V}}_n{}^k + h.o.t.$$

Finite Rotations - Linearization

$$\underline{\mathbf{u}} = \underline{\mathbf{u}}_I + \underline{\mathbf{u}}_{II} + h.o.t.$$

$$\underline{\mathbf{u}}_I = h_k\, \underline{\mathbf{u}}_k + \frac{r_3}{2}\, h_k\, {}^t a_k \left(-\theta_1{}^k\, {}^t\underline{\mathbf{V}}_2{}^k + \theta_2{}^k\, {}^t\underline{\mathbf{V}}_1{}^k\right)$$

$$\underline{\mathbf{u}}_{II} = -\frac{r_3}{4}\, h_k\, {}^t a_k \left[(\theta_1{}^k)^2 + (\theta_2{}^k)^2\right]\, {}^t\underline{\mathbf{V}}_n{}^k$$

$${}^{t+\Delta t}\underline{\tilde{\mathbf{g}}}_i = {}^t\underline{\tilde{\mathbf{g}}}_i + \frac{\partial \underline{\mathbf{u}}}{\partial r_i} \qquad (i = 1, 2, 3)$$

$${}^{t+\Delta t}_{\circ}\tilde{\varepsilon}_{ij} = \frac{1}{2}\left[{}^{t+\Delta t}\underline{\tilde{\mathbf{g}}}_i \cdot {}^{t+\Delta t}\underline{\tilde{\mathbf{g}}}_j - {}^{\circ}\underline{\tilde{\mathbf{g}}}_i \cdot {}^{\circ}\underline{\tilde{\mathbf{g}}}_j\right]$$

$${}^{t+\Delta t}_{\circ}\tilde{\varepsilon}_{ij} = {}^{t}_{\circ}\tilde{\varepsilon}_{ij} + \underbrace{\frac{1}{2}\left[{}^t\underline{\tilde{\mathbf{g}}}_i \cdot \frac{\partial \underline{\mathbf{u}}}{\partial r_j} + {}^t\underline{\tilde{\mathbf{g}}}_j \cdot \frac{\partial \underline{\mathbf{u}}}{\partial r_i} + \frac{\partial \underline{\mathbf{u}}}{\partial r_i} \cdot \frac{\partial \underline{\mathbf{u}}}{\partial r_j}\right]}_{{}_{\circ}\tilde{\varepsilon}_{ij} = {}_{\circ}\tilde{e}_{ij} + {}_{\circ}\tilde{\eta}_{ij}}$$

Finite Rotations - Linearization

$$ {}_{\circ}\tilde{e}_{ij} = \frac{1}{2}\left[{}^{t}\underline{\tilde{\mathbf{g}}}_{i} \cdot \frac{\partial \underline{\mathbf{u}}_{I}}{\partial r_{j}} + {}^{t}\underline{\tilde{\mathbf{g}}}_{j} \cdot \frac{\partial \underline{\mathbf{u}}_{I}}{\partial r_{i}} \right] $$

$$ {}_{\circ}\tilde{\eta}_{ij} = \frac{1}{2}\left[{}^{t}\underline{\tilde{\mathbf{g}}}_{i} \cdot \frac{\partial \underline{\mathbf{u}}_{II}}{\partial r_{j}} + {}^{t}\underline{\tilde{\mathbf{g}}}_{j} \cdot \frac{\partial \underline{\mathbf{u}}_{II}}{\partial r_{i}} + \frac{\partial \underline{\mathbf{u}}_{I}}{\partial r_{i}} \cdot \frac{\partial \underline{\mathbf{u}}_{I}}{\partial r_{j}} \right] $$

MITC4 – Finite Strains

$$ {}^{o}\underline{x}(r,s,t) = h_k(r,s)\ {}^{o}\underline{x}_k + \frac{t}{2}\ {}^{o}\underline{d}\ a $$

$$ {}^{o}\underline{d} = \frac{h_k(r,s)\ {}^{o}\underline{V}_n^k}{\left\| h_k(r,s)\ {}^{o}\underline{V}_n^k \right\|} $$

Gebhardt H. and Schweizerhof K. (1993), "Interpolation of curved shell geometries by low order finite elements - Errors and modifications", Int. J. Numerical Methods in Engng., vol. 36, pp.287-302.

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Introduce thickness stretching

$$ {}^{\tau}\underline{x}(r,s,t) = h_k(r,s)\ {}^{\tau}\underline{x}_k + \frac{t}{2}\ \left({}^{\tau}\lambda_o + {}^{\tau}\lambda_1 t\right)\ {}^{\tau}\underline{d}\ a $$

$$ {}^{\tau}\underline{d} = \frac{h_k(r,s)\ {}^{\tau}\underline{V}_n^k}{\left\| h_k(r,s)\ {}^{\tau}\underline{V}_n^k \right\|} $$

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$$\underline{u} = {}^{\tau+\Delta\tau}\underline{x} - {}^{\tau}\underline{x}$$

$$\begin{aligned}\underline{u}(r,s,t) = \; & h_k(r,s)\,\underline{u}_k \\ & + \frac{t}{2}\,a\,\left({}^{\tau}\lambda_o + \Delta\lambda_o + {}^{\tau}\lambda_1\,t + \Delta\lambda_1\,t\right)\,\frac{h_k(r,s)\;{}^{\tau+\Delta\tau}\underline{V}_n^k}{\left|\left|h_k(r,s)\;{}^{\tau+\Delta\tau}\underline{V}_n^k\right|\right|} \\ & - \frac{t}{2}\,a\,\left({}^{\tau}\lambda_o + {}^{\tau}\lambda_1 t\right)\,\frac{h_k(r,s)\;{}^{\tau}\underline{V}_n^k}{\left|\left|h_k(r,s)\;{}^{\tau}\underline{V}_n^k\right|\right|}\;.\end{aligned}$$

In the above, ${}^{\tau+\Delta\tau}\lambda_o = {}^{\tau}\lambda_o + \Delta\lambda_o$ and ${}^{\tau+\Delta\tau}\lambda_1 = {}^{\tau}\lambda_1 + \Delta\lambda_1$.

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For the director vector rotations we can write [35],

$$ {}^{\tau+\Delta\tau}\underline{V}_n^k = {}_{\tau}^{\tau+\Delta\tau}\underline{\underline{R}} \cdot {}^{\tau}\underline{V}_n^k $$

with,

$$ {}_{\tau}^{\tau+\Delta\tau}\underline{\underline{R}} = \underline{\underline{I}}_3 + \frac{\sin\theta_k}{\theta_k}\underline{\underline{\Theta}}^k + \frac{1}{2}\left(\frac{\sin(\theta_k/2)}{(\theta_k/2)}\right)^2 \left(\underline{\underline{\Theta}}^k\right)^2 $$

being $\underline{\underline{I}}_3$ is the second order unit tensor.

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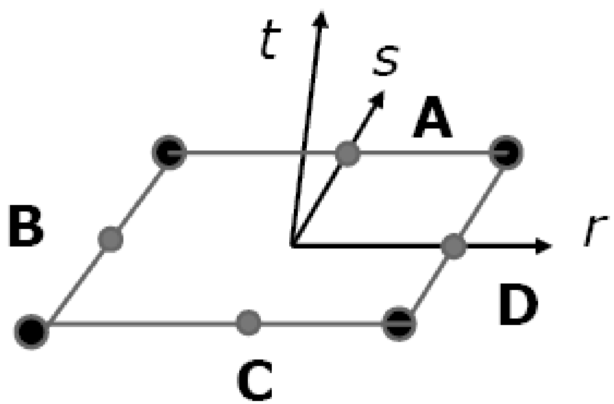
$$\boxed{{}^{\tau}_{o}\underline{\underline{H}} = {}^{\tau}_{o}\widetilde{H}_{ij}\; {}^{o}\underline{\widetilde{g}}^{i}\; {}^{o}\underline{\widetilde{g}}^{j}}$$



$$
\begin{aligned}
{}^{\tau}_{o}\widetilde{H}_{rr} &= \left[{}^{\tau}_{o}\widetilde{H}_{rr}\right]^{DI} \\
{}^{\tau}_{o}\widetilde{H}_{ss} &= \left[{}^{\tau}_{o}\widetilde{H}_{ss}\right]^{DI} \\
{}^{\tau}_{o}\widetilde{H}_{rs} &= \left[{}^{\tau}_{o}\widetilde{H}_{rs}\right]^{DI}
\end{aligned}
$$

$${}^{\tau}_{o}\widetilde{H}_{tt} = \left[{}^{\tau}_{o}\widetilde{H}_{tt}\right]^{DI}$$

$$
\begin{aligned}
{}^{\tau}_{o}\widetilde{H}_{rt} &= \frac{1}{2}\,(1+s)\,\left[{}^{\tau}_{o}\widetilde{H}_{rt}\right]^{DI}_{A} + \frac{1}{2}\,(1-s)\,\left[{}^{\tau}_{o}\widetilde{H}_{rt}\right]^{DI}_{C} \\
{}^{\tau}_{o}\widetilde{H}_{st} &= \frac{1}{2}\,(1+r)\,\left[{}^{\tau}_{o}\widetilde{H}_{st}\right]^{DI}_{D} + \frac{1}{2}\,(1-r)\,\left[{}^{\tau}_{o}\widetilde{H}_{st}\right]^{DI}_{B}
\end{aligned}
$$

MITC4 – Finite Strains

Use 3D constitutive relations

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