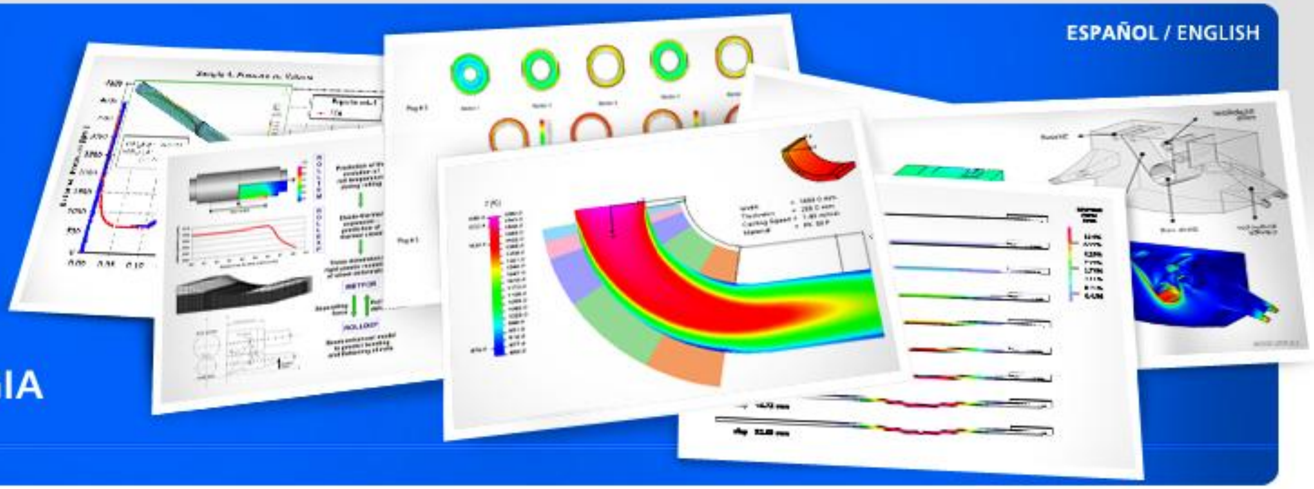
SIM&TEC

Simulación y Tecnología

Simulation and Technology

ESPAÑOL / ENGLISH



DE LA CIENCIA

A LA TECNOLOGIA

FEM in Solid Mechanics
Part I

Eduardo Dvorkin

Contents

Part 1

- ❑ Linear and nonlinear problems in solid mechanics
- ❑ Fundamental equations:
  - ✓ Kinematics
  - ✓ The stress tensor
  - ✓ Equilibrium
  - ✓ Constitutive relations
- ❑ The principle of virtual work

Part 2

- ❑ FEM in solid mechanics
- ❑ Elasto-plasticity
- ❑ Structural elements
- ❑ Nonlinear problems- Collapse
- ❑ Dynamic problems

Linear and nonlinear problems

Material nonlinearities

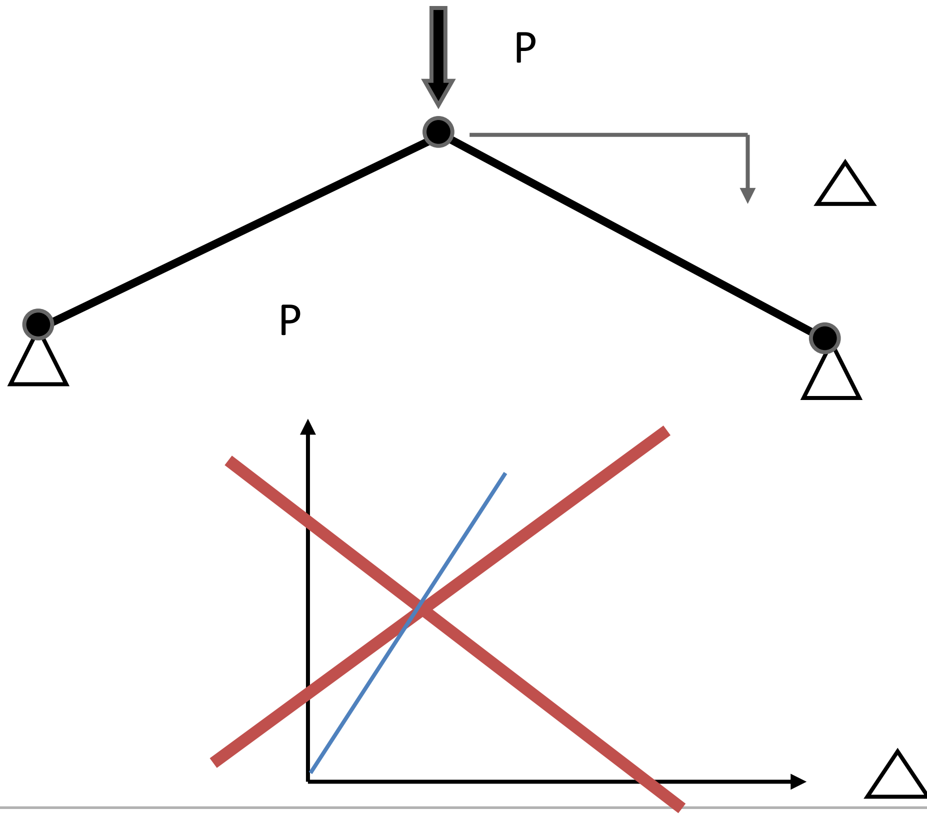
- Plasticity
- Viscoplasticity, creep
- Fracturing materials

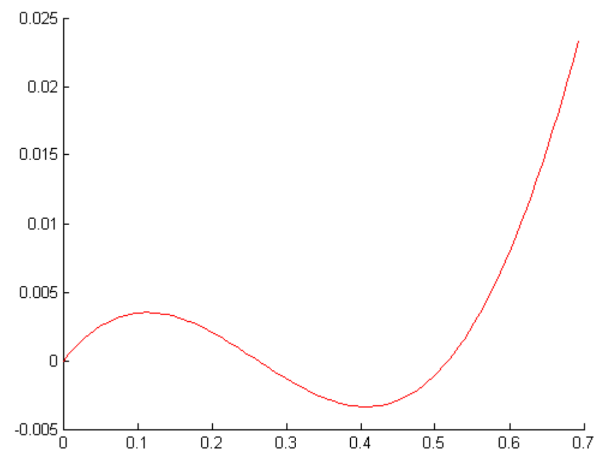
Geometrical nonlinearities

- Contact problems
- Equilibrium in deformed configuration
- Finite strains

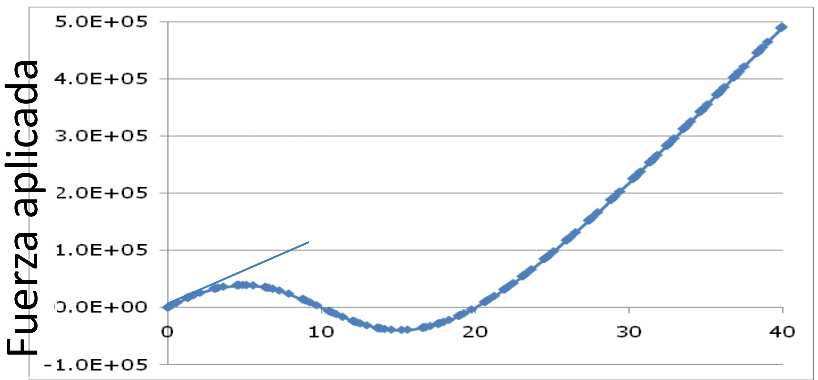
Linear and nonlinear problems

Geometrical nonlinearities:
Equilibrium in deformed configuration

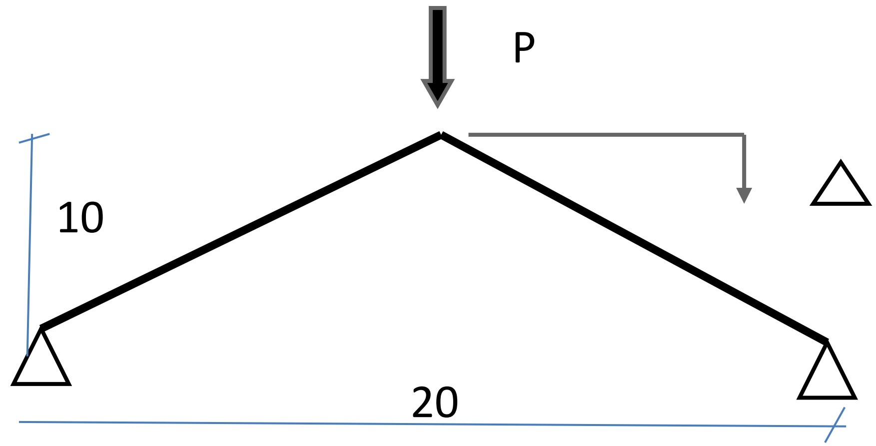




No linealidad Geométrica



Desplazamiento vertical

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              E       =    21000  NU      = 0.3

EGROUP TRUSS NAME = 1,
 DISPLACEMENTS     = LARGE,
 MATERIAL         = 1,
 INT           = 2,
 area         = 10.
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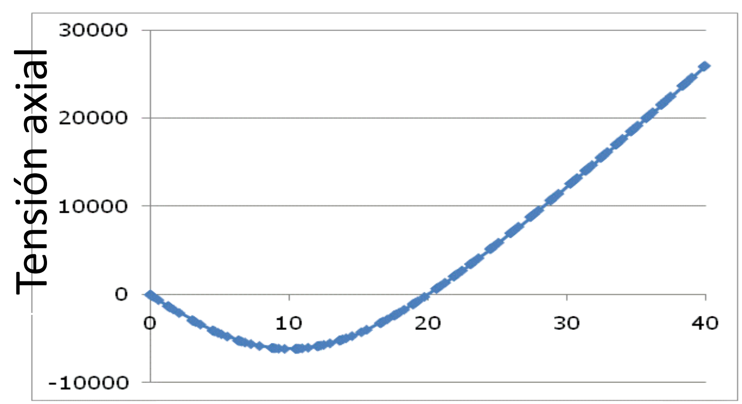
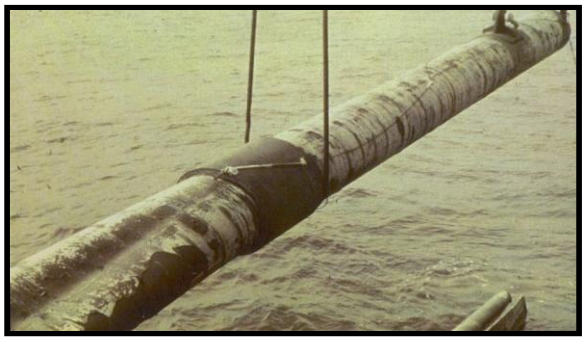


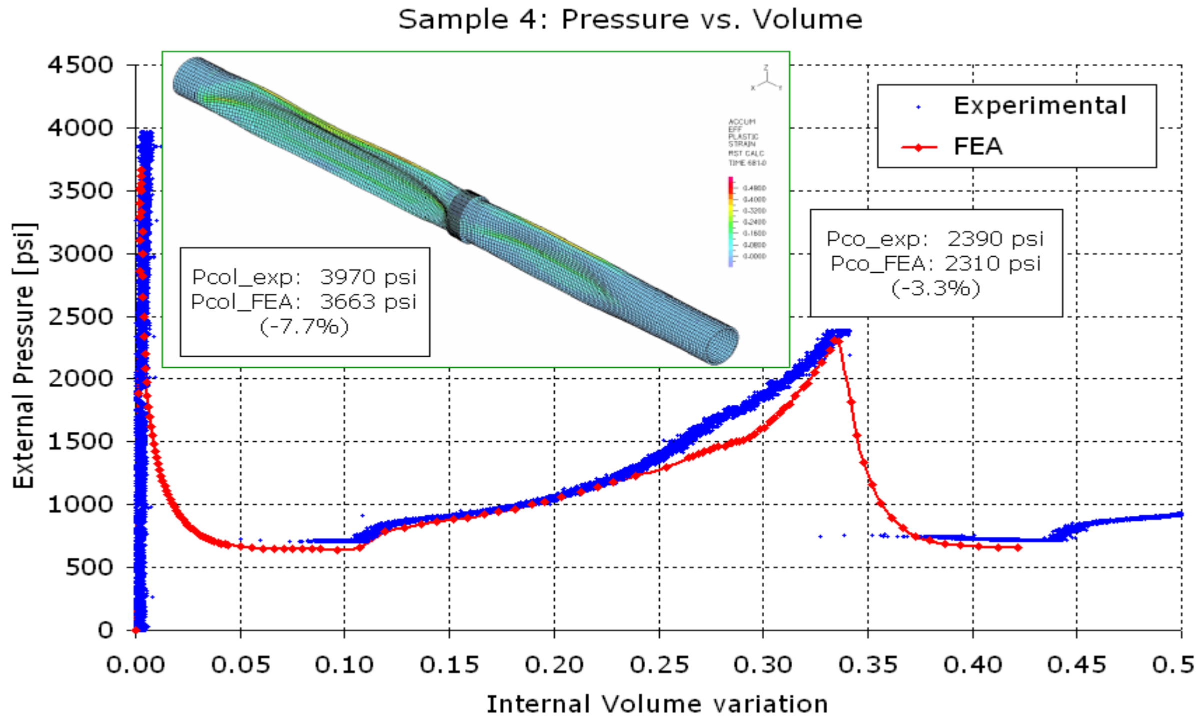
Desplazamiento vertical

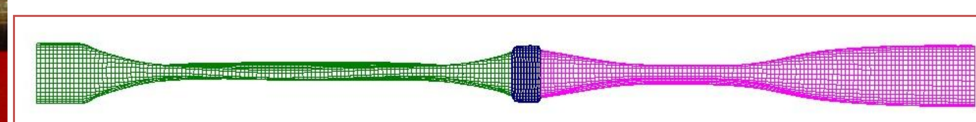
Linear and nonlinear problems

Material + Geometrical nonlinearities + Contact: Collapse



Buckle arrestors

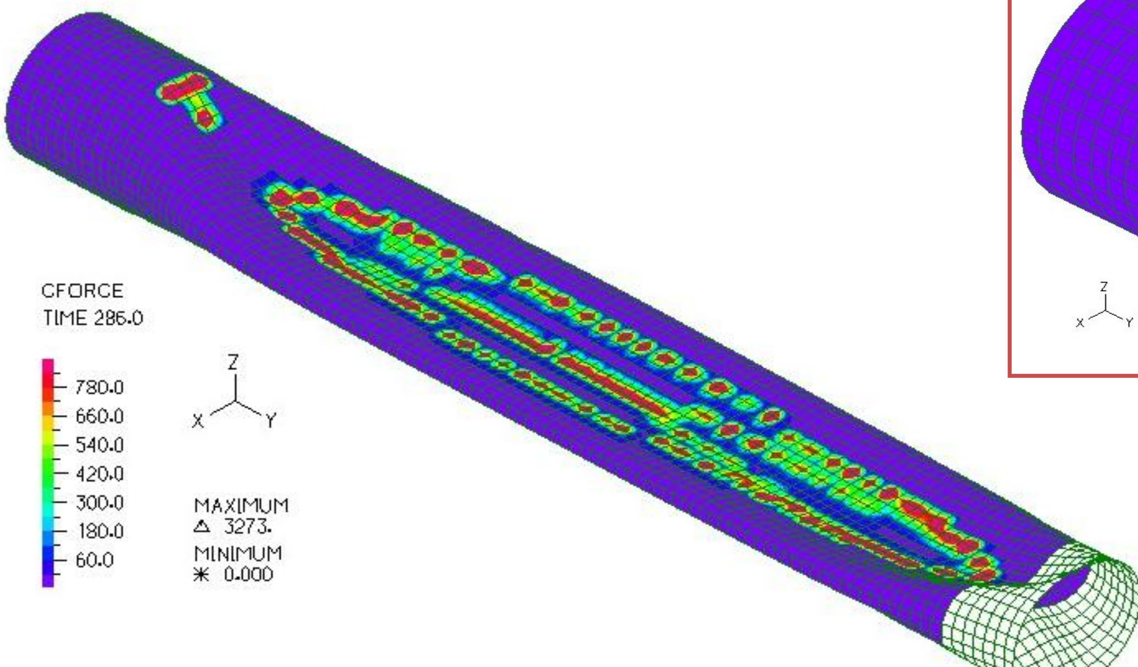


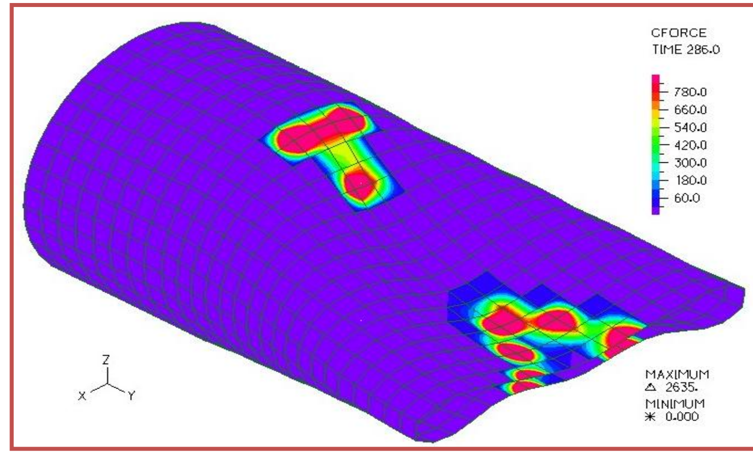


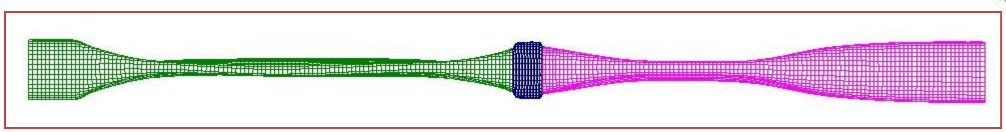
Linear and nonlinear problems

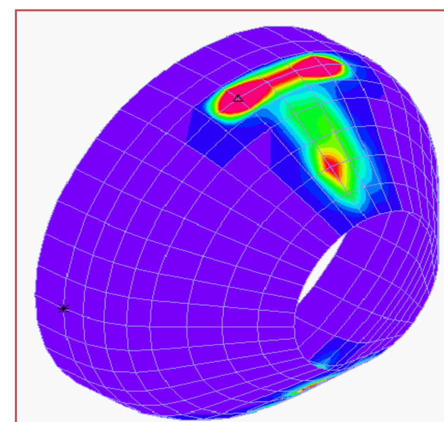
Material + Geometrical nonlinearities + Contact: Collapse

Contact forces



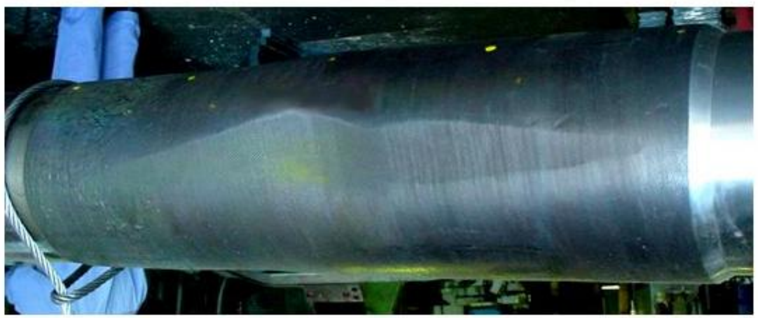


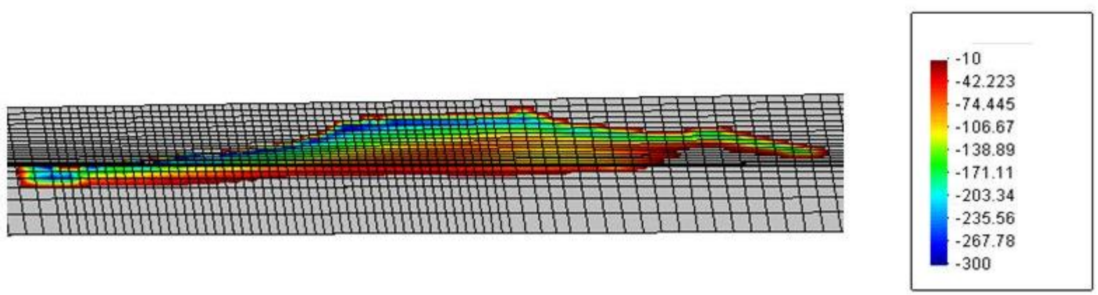




Linear and nonlinear problems

Material + Geometrical nonlinearities + Contact



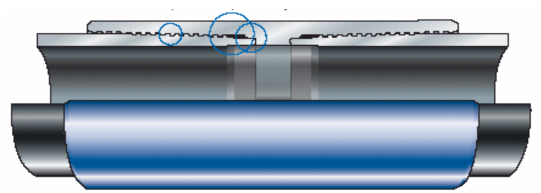


Contact Pressure [Mpa]

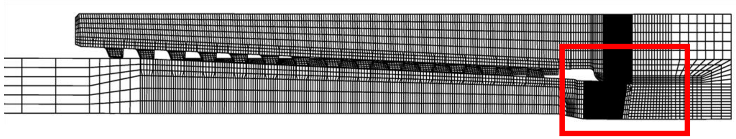
Linear and nonlinear problems

Material + Geometrical nonlinearities + Contact

OCTG connections

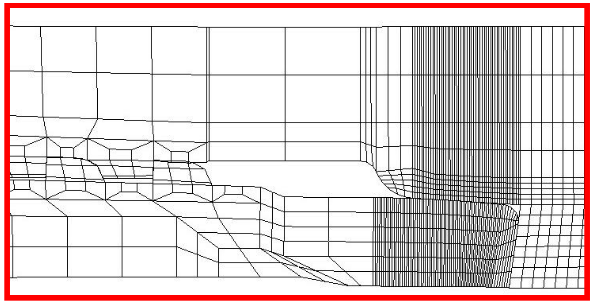


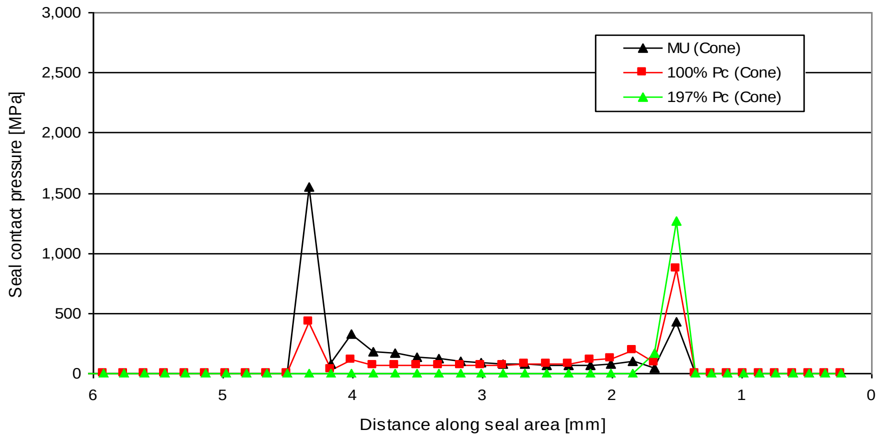
Structural Analysis

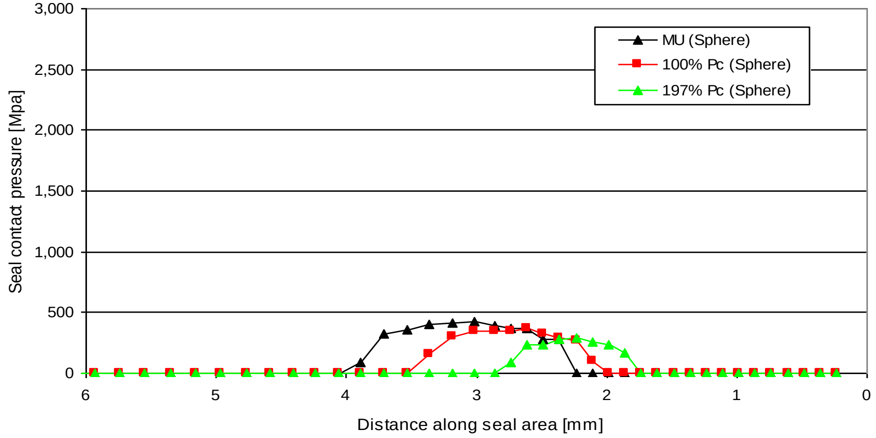


Detail of

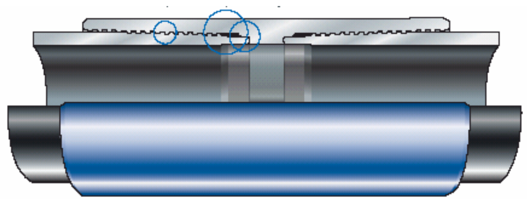
the seal area







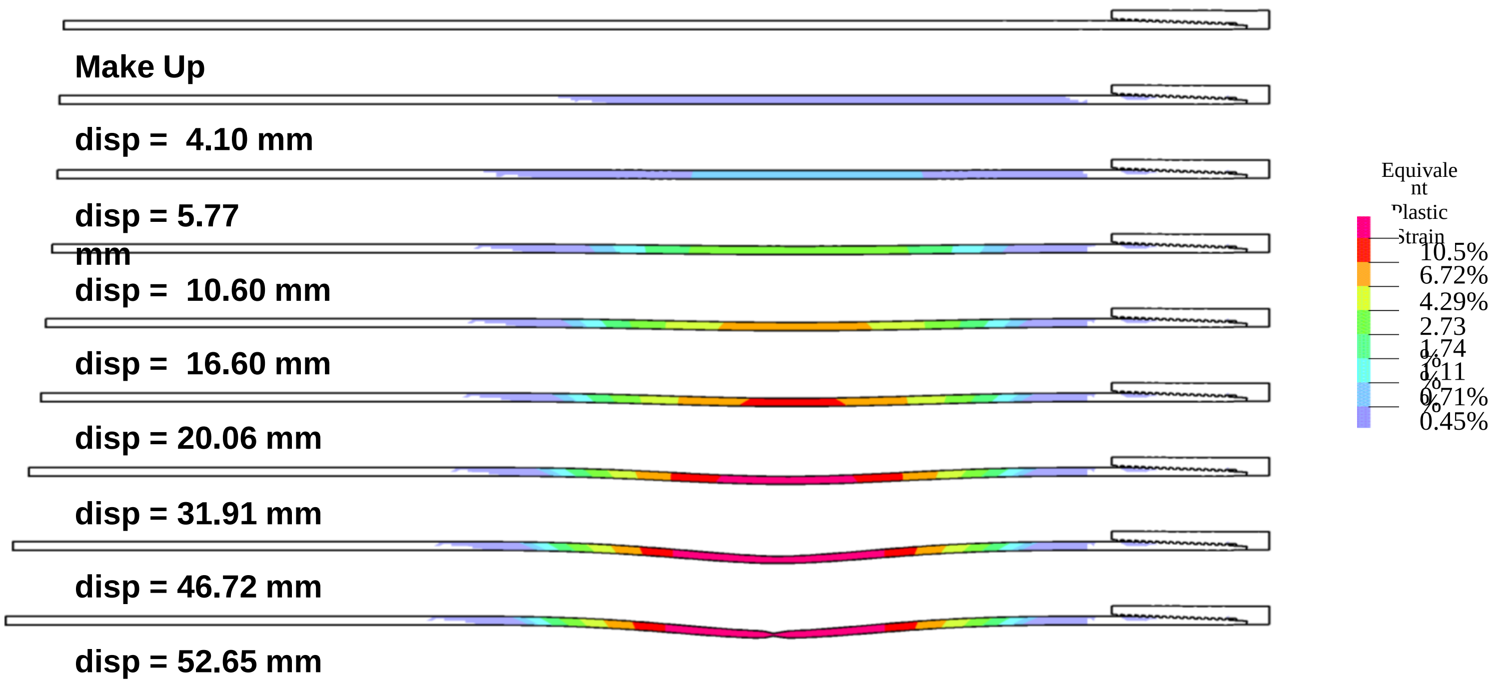
Linear and nonlinear problems



Material + Geometrical nonlinearities + Contact

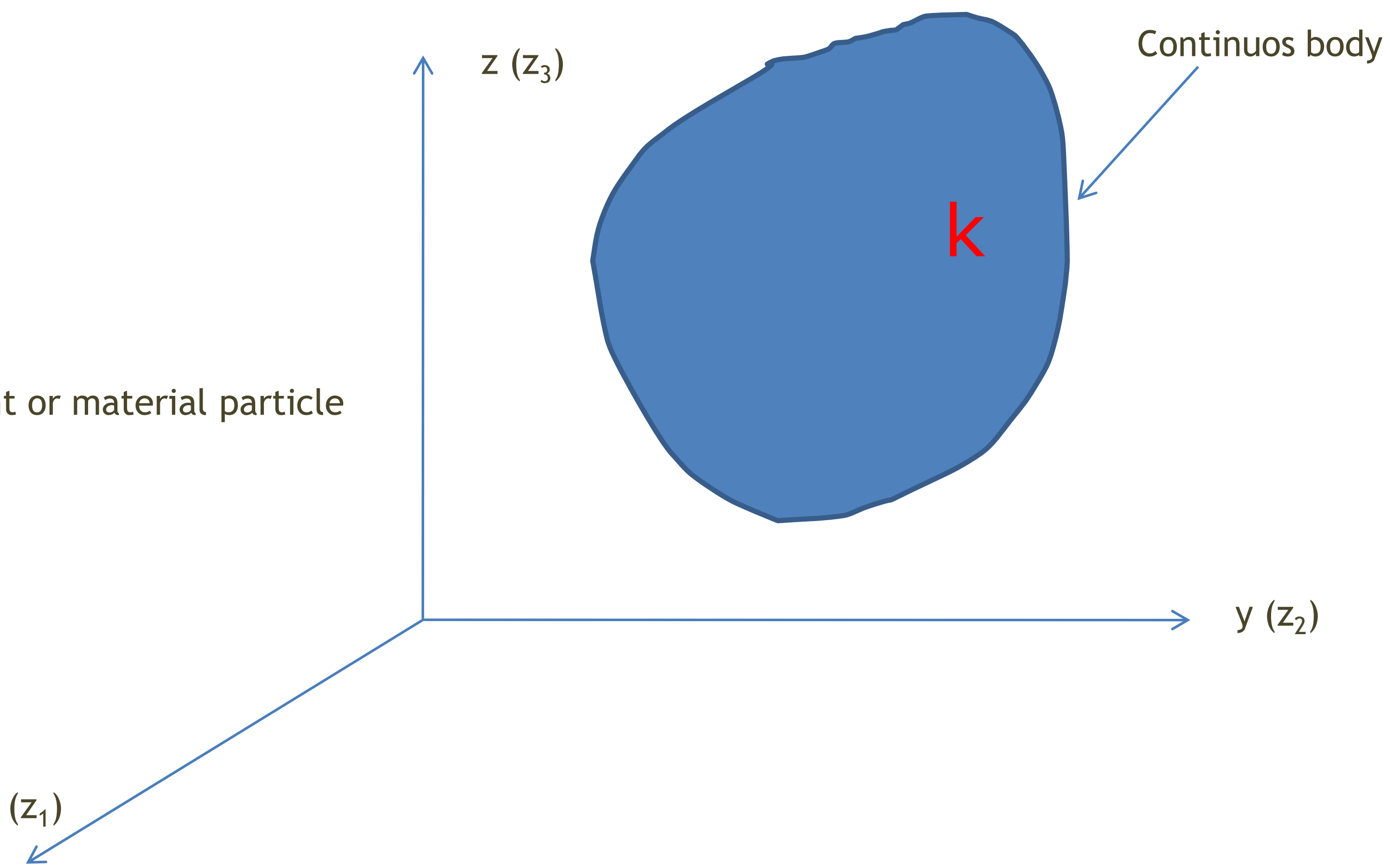
Finite elasto – plastic strains

OCTG connections: Failure analysis

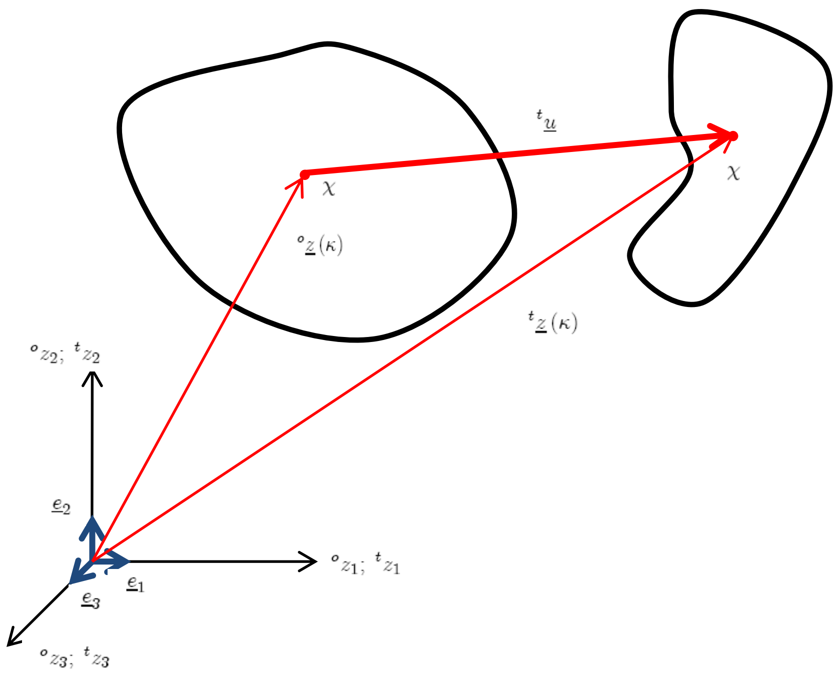


Cartesian Coordinate System

k: point or material particle



Kinematics of the Continuous Media



Lagrangian description

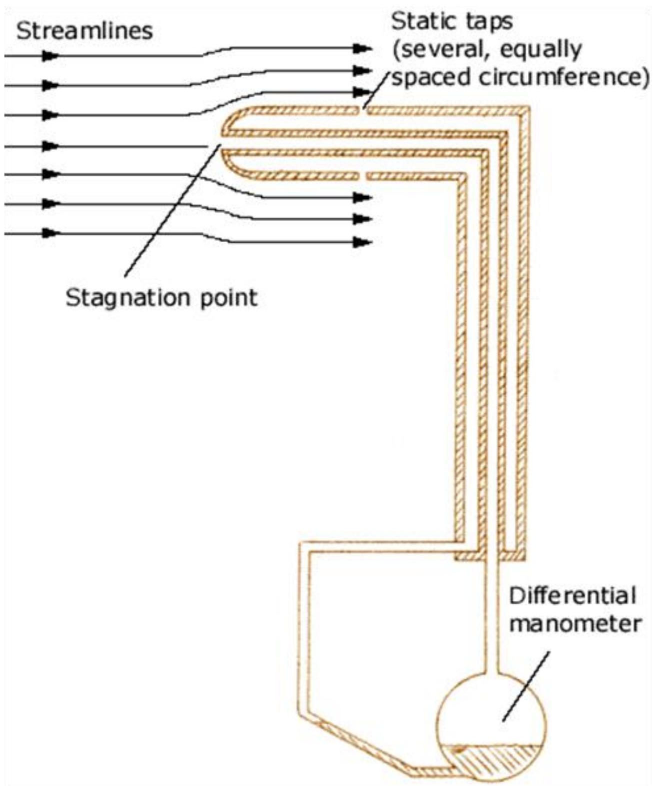
$$^t\underline{v} = \ ^t\underline{v}(\ ^o\underline{z}, t)$$

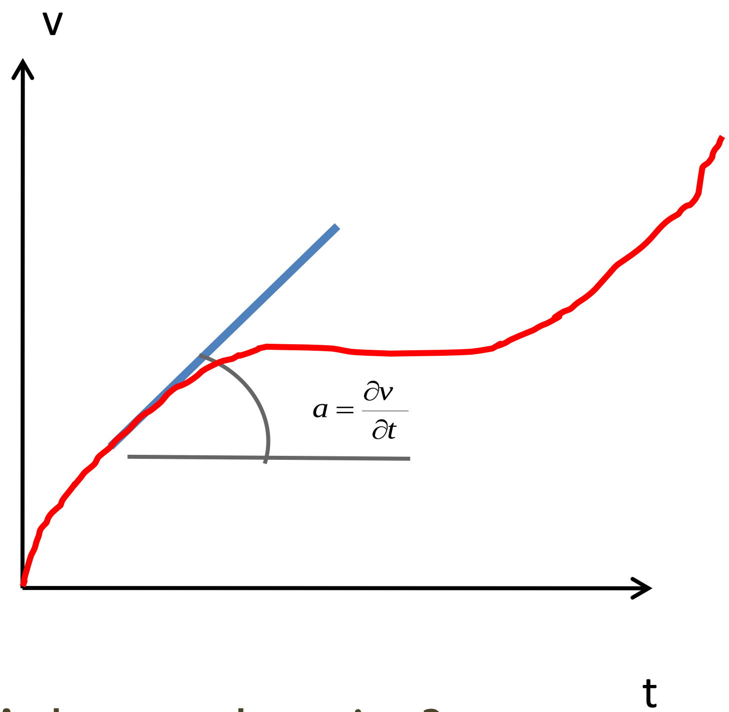
Eulerian description

$$^t\underline{v} = \ ^t\underline{v}(\ ^t\underline{z}, t)$$

Pop Quiz # 2

Eulerian formulation





Is “a” the acceleration?

NO

$$ {}^t\underline{a} = \frac{\partial\, {}^t\underline{v}({}^t\underline{z}, t)}{\partial\, t} + {}^t v_i \left(\frac{\partial\, {}^t v_i}{\partial\, {}^t z_j} \right) \underline{e}_j $$

Kinematics of the Continuous Media

$$\begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_{xx} & 2\varepsilon_{xy} & 2\varepsilon_{xz} \\ 2\varepsilon_{yx} & \varepsilon_{yy} & 2\varepsilon_{yz} \\ 2\varepsilon_{zx} & 2\varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{xx} & 2\varepsilon_{xy} & 2\varepsilon_{xz} \\ 2\varepsilon_{yx} & \varepsilon_{yy} & 2\varepsilon_{yz} \\ 2\varepsilon_{zx} & 2\varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \Rightarrow \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix}$$

Kinematics of the Continuous Media

The second transformation is only possible if the **compatibility equations** are fulfilled

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x\,\partial y} = 0$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} - 2\frac{\partial^2 \varepsilon_{yz}}{\partial y\,\partial z} = 0$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} - 2\frac{\partial^2 \varepsilon_{zx}}{\partial z\,\partial x} = 0$$

$$-\frac{\partial^2 \varepsilon_{xx}}{\partial y\,\partial z} + \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) = 0$$

$$-\frac{\partial^2 \varepsilon_{yy}}{\partial z\,\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) = 0$$

$$-\frac{\partial^2 \varepsilon_{zz}}{\partial x\,\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z}\right) = 0$$

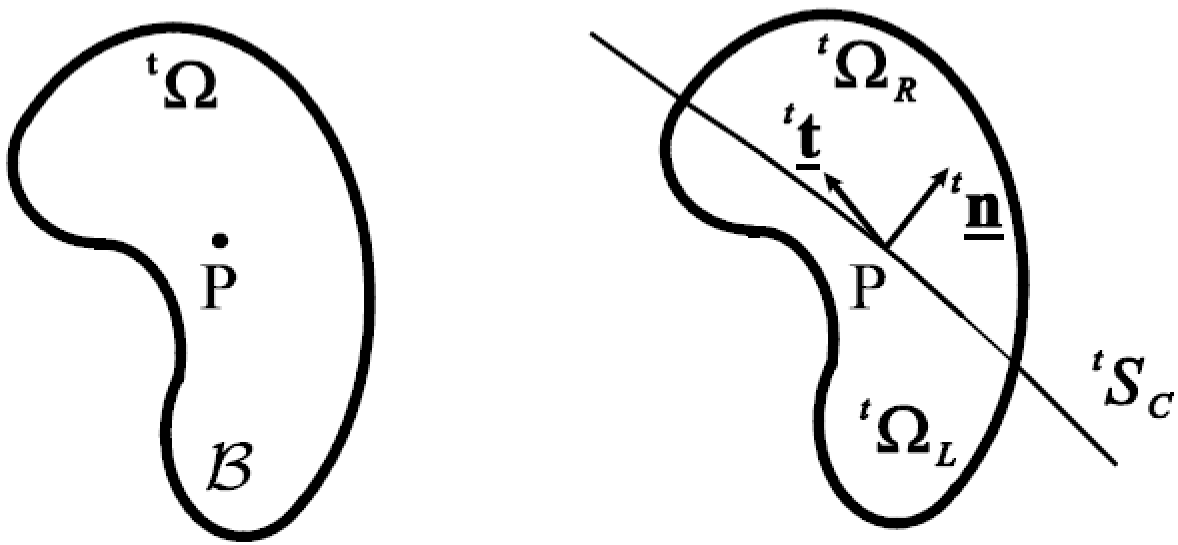
The Cauchy Stress Tensor

(a) Particle P inside the body $\mathcal{B}$

(b) Section of the body $\mathcal{B}$ with a surface through P

$${}^t\Omega_L \cap {}^t\Omega_R = 0$$

$${}^t\Omega_L \cup {}^t\Omega_R = {}^t\Omega$$



The Cauchy Stress Tensor

Considering on the surface ${}^{t}S_c$ an area ${}^{t}\Delta S$ around P, the set of external forces acting on ${}^{t}\Delta S$ can be reduced to a force ${}^{t}\Delta\underline{\mathbf{F}}$ through P and a moment ${}^{t}\Delta\underline{\mathbf{M}}_P$.

When ${}^{t}\Delta S \to 0$:

$$\lim_{{}^{t}\Delta S \to 0} \frac{{}^{t}\Delta\underline{\mathbf{F}}}{{}^{t}\Delta S} = {}^{t}\underline{\mathbf{t}} \tag{3.6a}$$

$$\lim_{{}^{t}\Delta S \to 0} \frac{{}^{t}\Delta\underline{\mathbf{M}}_P}{{}^{t}\Delta S} = \underline{\mathbf{0}} \tag{3.6b}$$

The vector ${}^{t}\underline{\mathbf{t}}$ is known in the literature as *traction*.

Equations(3.6a-3.6b) incorporate two fundamental hypotheses:

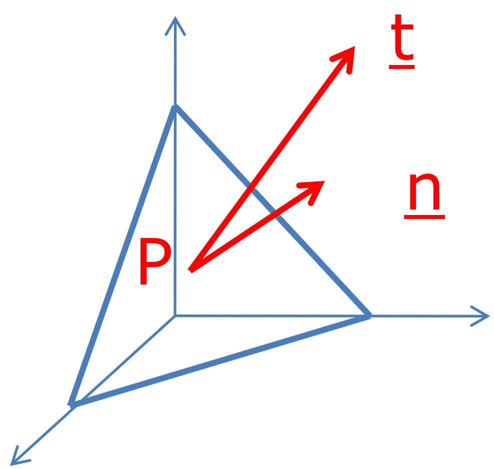
- The limit in Eq. (3.6a) *exists*. Therefore we exclude from the continuum mechanics field the consideration of concentrated forces (concentrated forces are also not physically possible).
- The condition in Eq. (3.6b) is a strong requirement in the classical formulation of continuum mechanics. There are alternative formulations that do not require the fulfillment of Eq. (3.6b) (e.g. the theory of polar media (Truesdell & Noll 1965, Malvern 1969).

The Cauchy Stress Tensor

Definition:

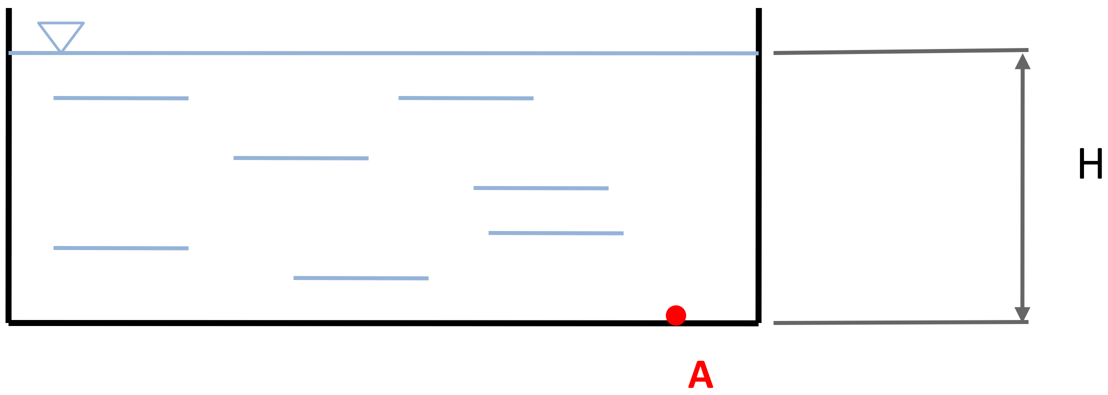
$$ {}^{t}t_i = {}^{t}n_j \, {}^{t}\sigma_{ji} \qquad \text{(i=1,2,3)} $$

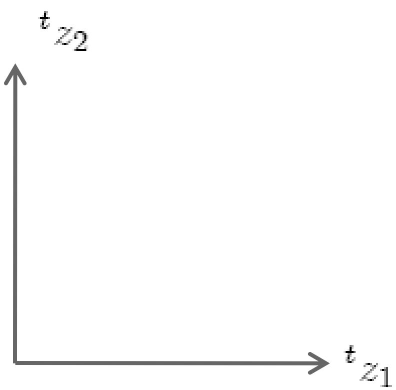
(Above we use the summation convention)



Pop Quiz # 3

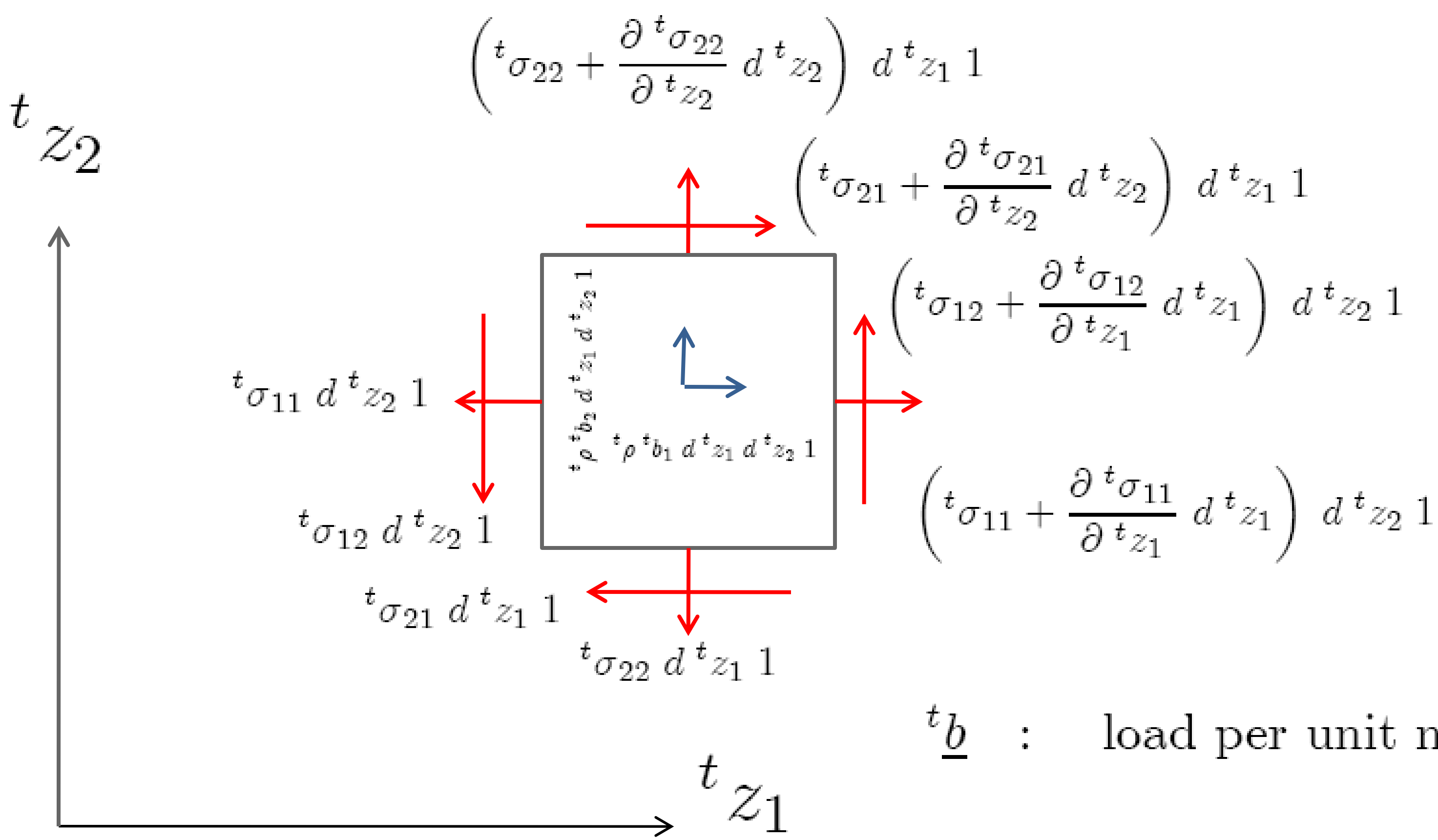
Water tank





$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}_A = \begin{bmatrix} -\rho g H & 0 & 0 \\ 0 & -\rho g H & 0 \\ 0 & 0 & -\rho g H \end{bmatrix}$$

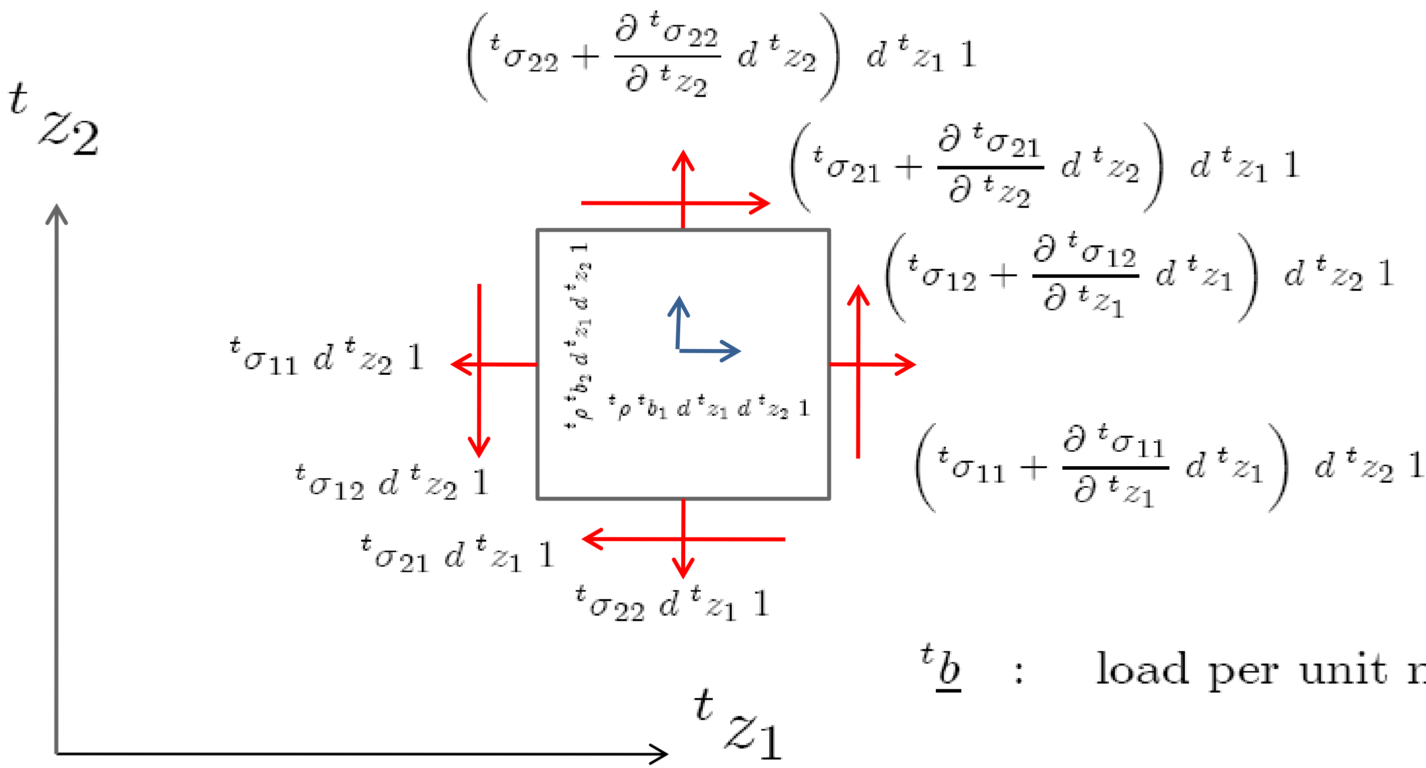
The Cauchy Stress Tensor



${}^t\underline{b}$: load per unit mass

The Cauchy Stress Tensor

Equilibrium in the deformed configuration



${}^t\underline{b}$: load per unit mass

The Cauchy Stress Tensor

Equilibrium in the deformed configuration

$$
\begin{aligned}
\frac{\partial\, {}^t\sigma_{11}}{\partial\, {}^t z_1} + \frac{\partial\, {}^t\sigma_{21}}{\partial\, {}^t z_2} + \frac{\partial\, {}^t\sigma_{31}}{\partial\, {}^t z_3} + {}^t\rho\, {}^t b_1 &= 0 \\
\frac{\partial\, {}^t\sigma_{12}}{\partial\, {}^t z_1} + \frac{\partial\, {}^t\sigma_{22}}{\partial\, {}^t z_2} + \frac{\partial\, {}^t\sigma_{32}}{\partial\, {}^t z_3} + {}^t\rho\, {}^t b_2 &= 0 \\
\frac{\partial\, {}^t\sigma_{13}}{\partial\, {}^t z_1} + \frac{\partial\, {}^t\sigma_{23}}{\partial\, {}^t z_2} + \frac{\partial\, {}^t\sigma_{33}}{\partial\, {}^t z_3} + {}^t\rho\, {}^t b_3 &= 0
\end{aligned}
$$

$\underline{b}$ is the force per unit mass

In dynamic analyses include in $\underline{f}$ the inertia forces.

The Cauchy Stress Tensor

Torques equilibrium

$$ {}^{t}\sigma_{ji} = {}^{t}\sigma_{ji} $$ (symmetry)

The Cauchy Stress Tensor and Physic

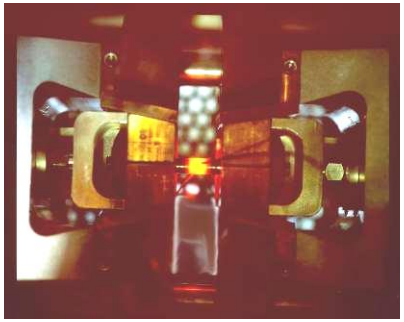
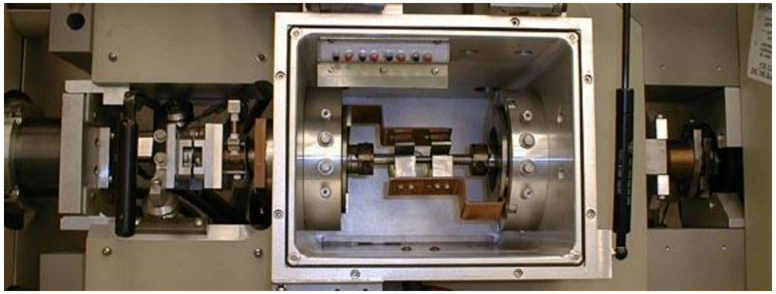
In nonlinear problems there are a number
of stress measures that are used during calculations:

- Kirchhoff stress tensor
- Second Piola-Kirchhoff stress tensor
- Biot stress tensor
- etc.

They are only mathematical tools.

The final results with significance for us
should be expressed in terms of
the Cauchy stress tensor.

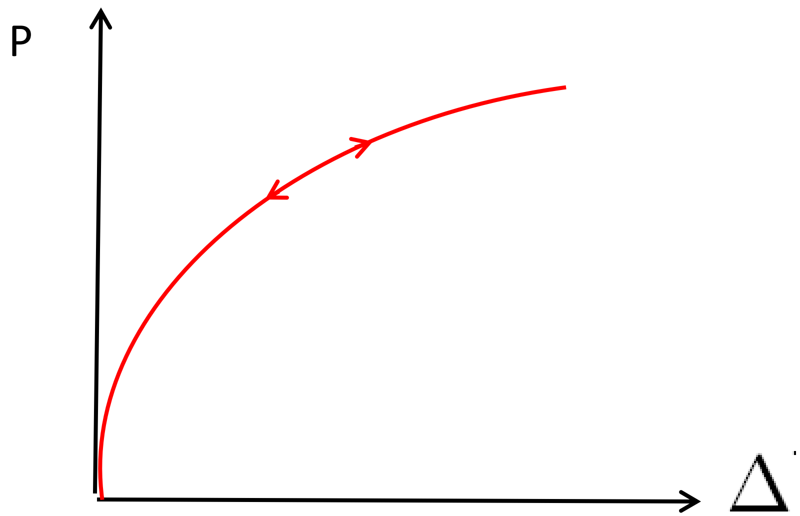
Constitutive Relations

$$[P, \Delta] \stackrel{calculations}{\longrightarrow} [\sigma, \varepsilon]$$

Phenomenological constitutive relations

Constitutive Relations



Elastic material

Constitutive Relations

HOOKE's LAW

Linear – elastic and isotropic materials

$$\begin{bmatrix} {}^t\sigma_{11} \\ {}^t\sigma_{22} \\ {}^t\sigma_{33} \\ {}^t\sigma_{12} \\ {}^t\sigma_{23} \\ {}^t\sigma_{31} \end{bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^t\varepsilon_{11} \\ {}^t\varepsilon_{22} \\ {}^t\varepsilon_{33} \\ {}^t\varepsilon_{12} \\ {}^t\varepsilon_{23} \\ {}^t\varepsilon_{31} \end{bmatrix}$$

$$\begin{aligned} E &: \text{ Young´s modulus} \\ E &> 0 \\ \nu &: \text{ Poisson's coefficient} \\ 0 &\leq \nu \leq 0.5 \\ E &= E(T) \\ \nu &= \nu(T) \end{aligned}$$

Constitutive Relations

Elasto-plastic materials

Ingredients:

- **Yield surface:** in the 3D stress space describes the locus of the points where the plastic behavior is initiated.
- **Flow rule:** describes the evolution of the plastic deformations.
- **Hardening law:** describes the evolution of the yield surface during the plastic deformation process.

Constitutive Relations

Elasto-plastic materials

In his experimental work, developed in the 1950s, Bridgman found that for metals, it can be assumed that the yield function is not affected by the confining hydrostatic pressure - at least for not very extreme hydrostatic pressures

${}^{t}s_{ij}$: deviatoric stress tensor

$$\begin{aligned} {}^{t}\sigma_{ij} &= {}^{t}s_{ij} + {}^{t}p\,\delta_{ij} \\ {}^{t}p &= \frac{1}{3}\left({}^{t}\sigma_{ij}\,\delta_{ij}\right) \\ {}^{t}s_{ij}\,\delta_{ij} &= 0 \end{aligned}$$

Constitutive Relations

Elasto-plastic materials

Von Mises yield function (metals)

$$
\begin{aligned}
{}^t f &= \frac{1}{2}\left({}^t s_{ij} - {}^t \alpha_{ij}\right)\left({}^t s_{ij} - {}^t \alpha_{ij}\right) - \frac{\left({}^t \sigma_y\right)^2}{3} \\
{}^t f &< 0 \text{ (elasticity)} \\
{}^t f &= 0 \text{ (plastic loading)} \\
{}^t \sigma_y &: \text{ tield stress at time "t"} \\
{}^t \alpha_{ij} &: \text{ back-stresses (kinematic hardening)}
\end{aligned}
$$

Constitutive Relations

Elasto-plastic materials

Von Mises yield function (metals)

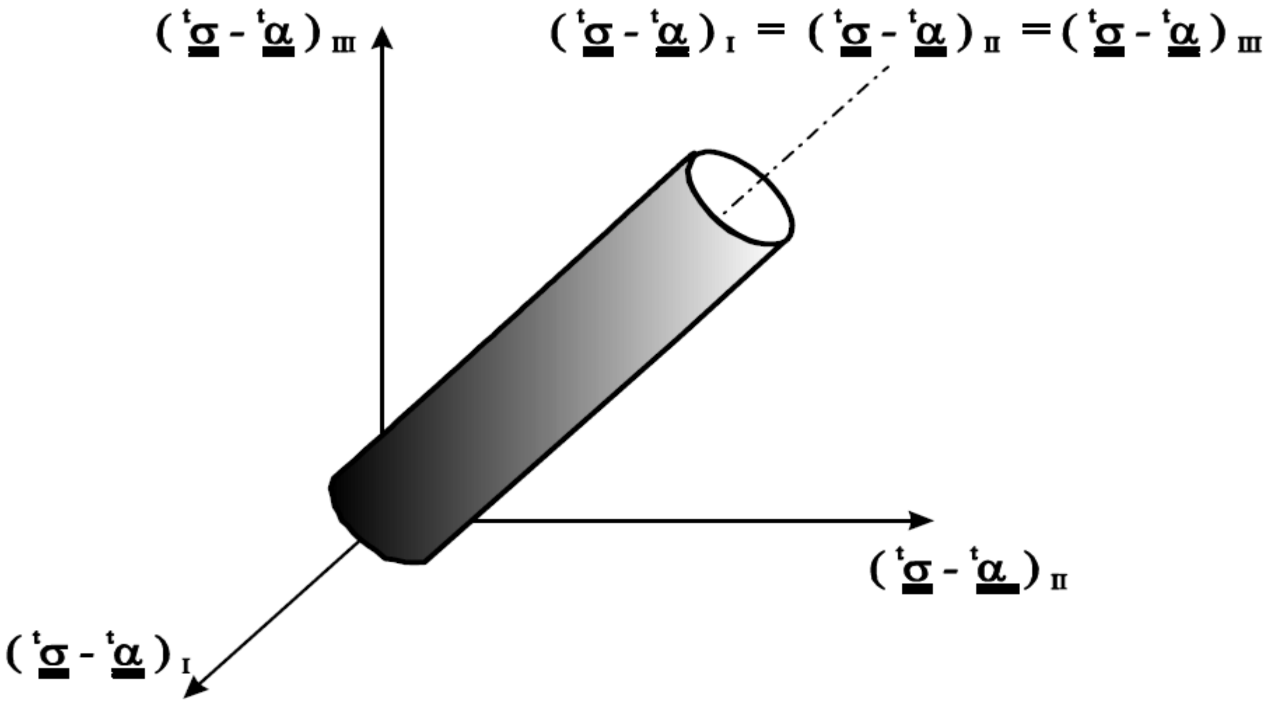


Fig. 5.5. Von Mises yield surface

Constitutive Relations

Elasto-plastic materials: The flow rule

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{E}_{ij} + \dot{\varepsilon}^{P}_{ij}$$

$$\dot{\varepsilon}^{P}_{ij} = \lambda \frac{\partial g}{\partial \sigma_{ij}}$$

g: plastic potential

$$For\ associated\ plasticity\,(metals)$$

$$g \equiv f$$

Constitutive Relations

Elasto-plastic materials: The flow rule

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{E}_{ij} + \dot{\varepsilon}^{P}_{ij}$$

$$\dot{\varepsilon}^{P}_{ij} = \lambda \frac{\partial g}{\partial \sigma_{ij}}$$

g: plastic potential

$$For\ associated\ plasticity\ (metals)$$

$$g \equiv f$$

Constitutive Relations

Elasto-plastic materials: The flow rule

For metals (von Mises + associated plasticity)

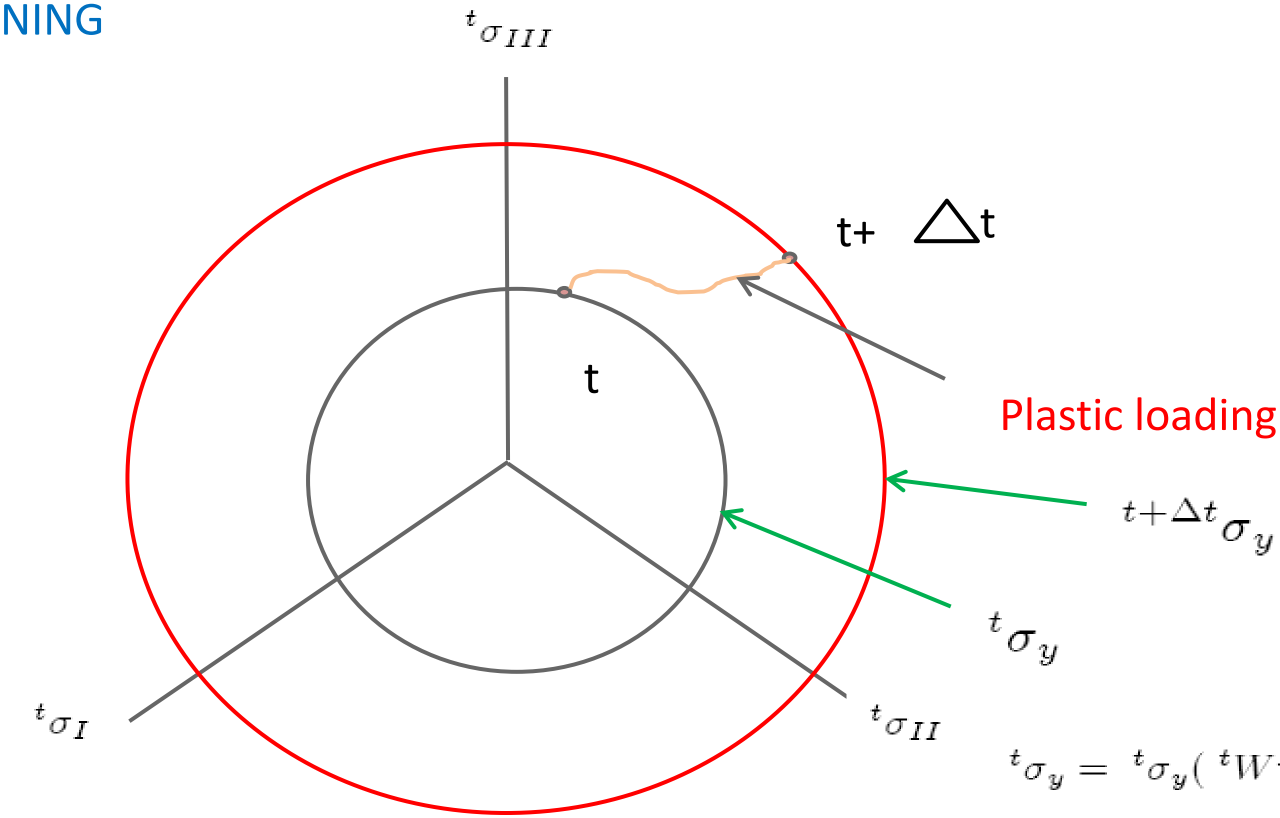
$$\dot{\varepsilon}^{P}_{xx} + \dot{\varepsilon}^{P}_{yy} + \dot{\varepsilon}^{P}_{zz} = 0$$

The plastic flow is incompressible

Constitutive Relations

Elasto-plastic materials:Hardening

ISOTROPIC HARDENING

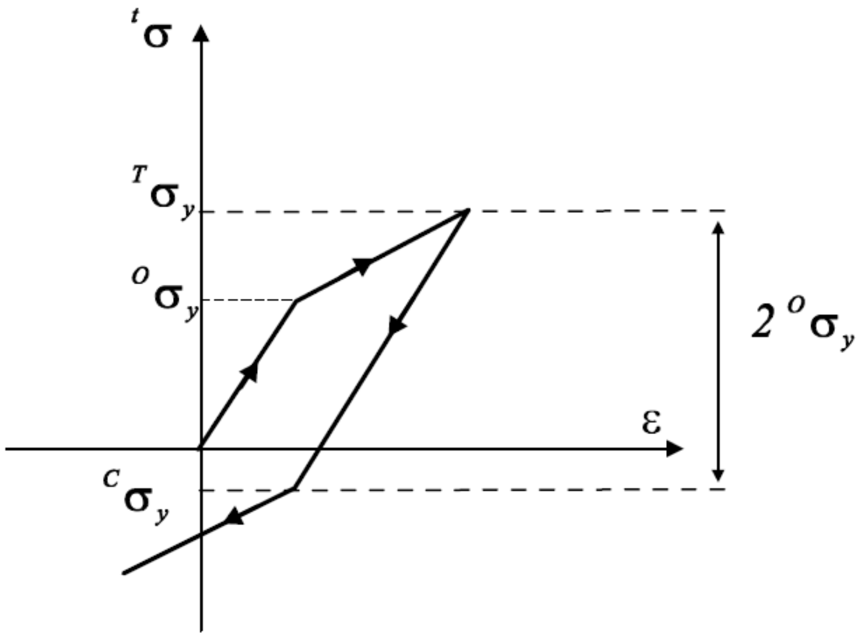


$$^t\sigma_y = \, ^t\sigma_y(\, ^tW^P)$$

Constitutive Relations

Elasto-plastic materials:Hardening

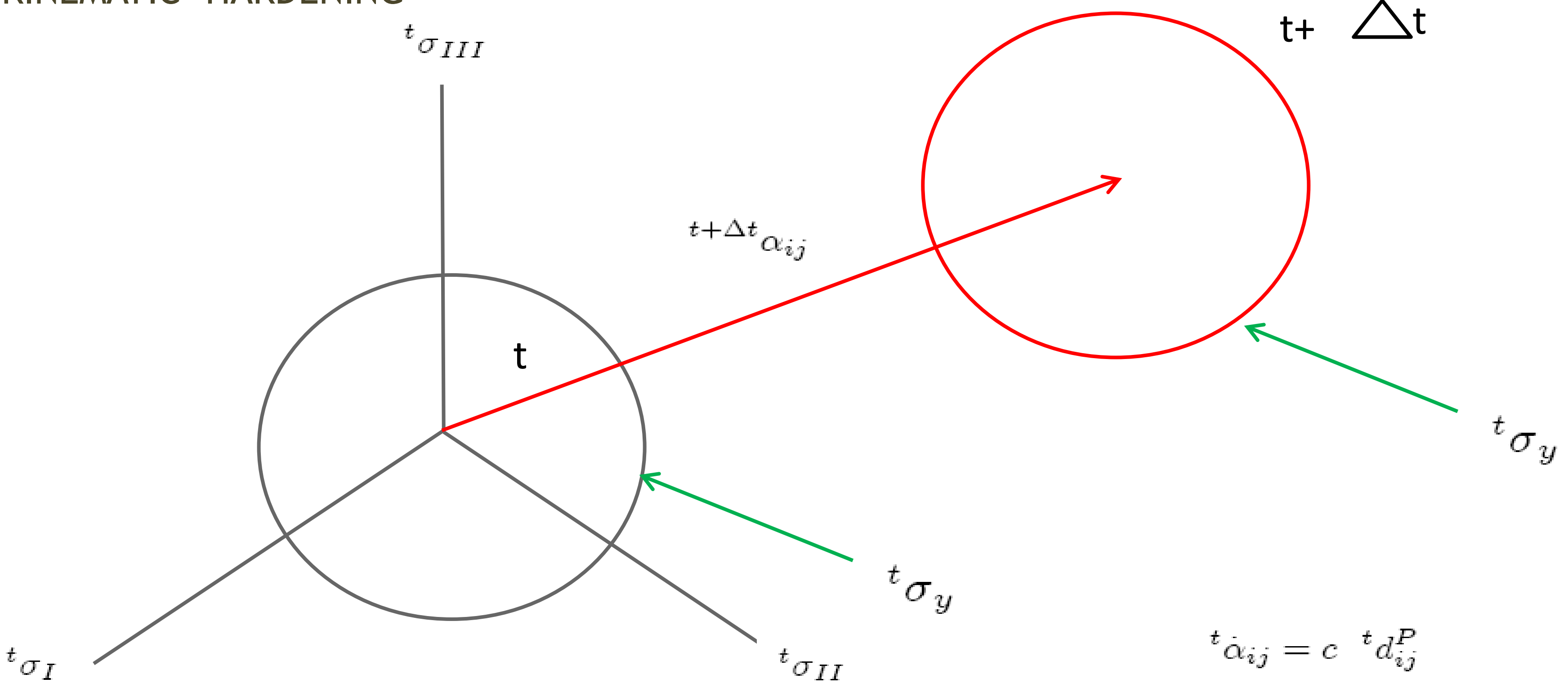
The isotropic hardening does not model
the Bauschinger effect (Cyclic loading / unloading)



Constitutive Relations

Elasto-plastic materials:Hardening

KINEMATIC HARDENING



$$ {}^{t}\dot{\alpha}_{ij} = c \; {}^{t}d^{P}_{ij} $$

Constitutive Relations

Viscoplasticity

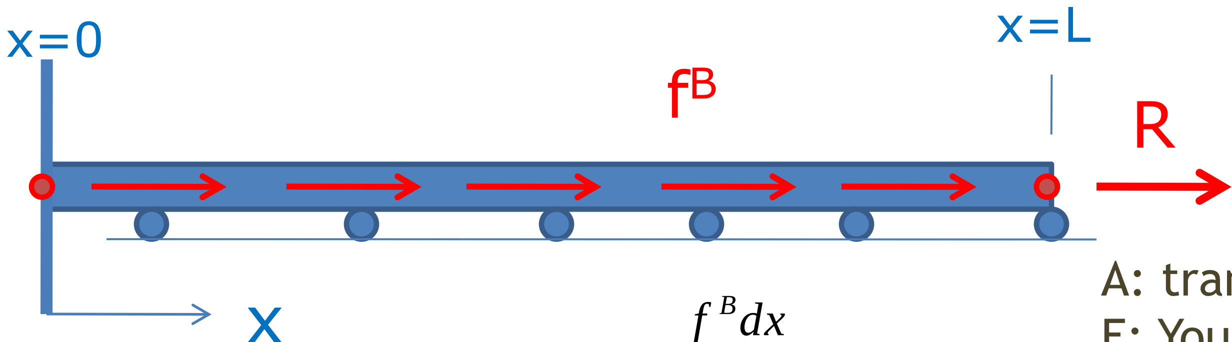
In plasticity: $$\sigma_y = \sigma_y(\bar{\varepsilon}, T)$$

We need viscoplasticity to model the experimental fact:

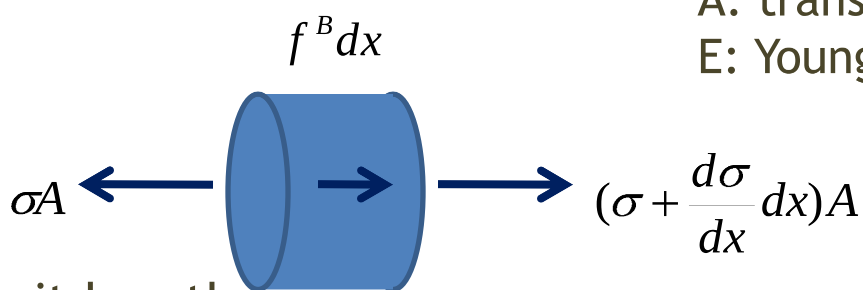
$$\sigma_y = \sigma_y(\bar{\varepsilon}, \dot{\bar{\varepsilon}}, T)$$

The principle of virtual work

A 1D problem



A: transversal area
E: Young's modulus



f^B : load per unit length
R : concentrated load

u(x) : unknown

The principle of virtual work

A 1D problem

$$u(0) = 0$$

Essential (rigid) boundary condition

$$EA\left[\frac{du}{dx}\right]_{x=L} = R$$

Natural boundary condition

The principle of virtual work

A 1D problem

Equilibrium:

$$A\frac{d\sigma}{dx}+f^{B}=0$$

Constitutive equation:

$$\sigma = E\varepsilon$$

Kinematic relation:

$$\varepsilon = \frac{du}{dx}$$

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The principle of virtual work

A 1D problem

At every point inside the bar we must fulfill:

$$AE\frac{d^2u}{dx^2}+f^B=0$$

δu(x) is an arbitrary function

δu(0)=0 (condition)

Hence,

$$\int_0^L\left(AE\frac{d^2u}{dx^2}+f^B\right)\delta u\,dx=0$$

The principle of virtual work

A 1D problem

Integrating by parts,

$$\int_0^L A\sigma\, \delta\varepsilon\, dx = \int_0^L f^B\, \delta u\, dx + R\delta u\Big|_{x=L}$$

Virtual work of internal forces = Virtual work of external forces

The principle of virtual work

Please notice that the PVW represents
Equilibrium
and NOT
Energy Conservation

The principle of virtual work

General 3D case

$$\int_{^tV} {}^t\sigma_{ij}\,\delta\varepsilon_{ij}\,d\,{}^tV = \int_{^tV} {}^tb_i\,\delta u_i\,d\,{}^tV + \int_{^tS_\sigma} {}^tt_i\,\delta u_i\,d\,{}^tS + {}^tF_i\,\delta u_i$$

b: Loads per unit volume
t: Loads per unit surface

Please notice that the integral is calculated at the deformed (unknown) configuration

The principle of virtual work

No material restriction (applies to any material)

No kinematics restriction (large or small strains)

No loads restriction (conservative or non-conservative)

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