Modeling of metal forming processes: Implementation of an iterative solver in the flow formulation

Dolores Demarco and Eduardo N. Dvorkin
Center for Industrial Research, FUDETEC
Av. Córdoba 320
1054, Buenos Aires, Argentina
E-mail: dvk@siderca.com
Fax: 54-11-4310-1000

June 26, 2001

Abstract

Rigid - viscoplastic models of metal forming processes are mathematically formulated as nonlinear Stokes problems: incompressible flows with variable viscosity, function of the point velocities. When discretizing those mathematical models using either u-p or velocity interpolated finite element formulations with the incompressibility imposed via penalty techniques, bad conditioned matrices are obtained. In this paper we discuss the implementation of an iterative solver for the efficient solution of the resulting equation systems.

keywords: metal forming; incompressibility; iterative solvers; conjugate gradients; preconditioning.

1 Introduction

The use of rigid - viscoplastic material models for the simulation of bulk metal forming processes (*flow formulation*) has proven to be a very efficient and reliable engineering tool and it is nowadays widely used for actual industrial applications [19]. This simulation technique leads to the mathematical description of incompressible flows in which the variable viscosity is a pointwise function of the velocity gradients (nonlinear Stokes problem). The discretization of this mathematical model is usually performed using either u-p or velocity interpolated finite element formulations with the incompressibility constraint imposed via penalty procedures. Therefore, the resulting stiffness matrices are badly conditioned [20]. If instead of a standard penalty method, an augmented Lagrangian procedure is used, the conditioning of the problem is only alleviated to

a certain degree, because it is still important to keep a relatively high penalty coefficient if the number of iterations to be used when solving the problem is going to remain within acceptable limits [13] [20] [7].

Our research group implemented the flow formulation via the pseudo - concentrations technique [17] [18]; in this implementation the bad conditioning of the resulting stiffness matrix is even worst than in the standard implementations of the flow formulation [4] - [12].

To get realistic simulations of industrial bulk metal forming processes using the finite element method, it is usually necessary to solve 3D models with a large number of degrees of freedom; these models are nonlinear and are therefore solved using iterative techniques such as the Newton-Raphson method; hence, a linear algebraic system of equations of the form $\underline{KU} = \underline{R}$, which represents for each iteration the linear approximation to the momentum conservation equations, needs to be solved a number of times. A large amount of computer memory allocation is necessary to store the nonzero numbers in the $\underline{K}$ matrix and also a large amount of CPU time is necessary to repetitively solve the linear algebraic equations system (the meaning of *"large"* is related to the available computational resources).

It is nowadays well known that the use of iterative solvers to solve a linear algebraic system of equations saves on the amount of required storage and also saves considerably CPU time [2]; however, in the case of the flow formulation the bad conditioning of the stiffness matrix introduces an important difficulty for the convergence of iterative solvers.

In the second section of this paper we briefly review the flow formulation and the main features of the algebraic system of equations that we get when linearizing the discretized problem. In the third section we discuss the performance that we can expect, when solving that system, from the preconditioned Conjugate Gradient Method; and in the fourth section we specifically discuss our implementation of the Canga - Becker preconditioner for the Conjugate Gradient Method [3]. This preconditioner was developed to overcome the bad conditioning coming from the incompressibility constraint and we extend it to the case of non-homogeneous mechanical behavior due to non-homogeneous strain and strain rate distributions.

In the fifth section we review the Jacobi preconditioner that we use combined with our implementation of the Canga - Becker preconditioner and finally in the sixth section we present numerical examples that illustrate on the performance of the implemented iterative solver.

2 The flow formulation

Considering a continuum in a $t-$spatial configuration we can determine the values of the ${}^{t}s_{ij}$, Cartesian components of the deviatoric Cauchy stress tensor and the values of the ${}^{t}\dot{\varepsilon}_{ij}$, Cartesian components of the strain rate tensor. We define the equivalent strain rate as

$$ {}^t\dot{\overline{\varepsilon}} = \sqrt{\frac{2}{3} {}^t\dot{\varepsilon}_{ij} \, {}^t\dot{\varepsilon}_{ij}} \, . \tag{1} $$

When modeling the behavior of metals the usual approach is to use an associated viscoplastic model with the von Mises yield function [16]; in these models the material strain hardening is described using the accumulated effective viscoplastic strain, $\overline{\varepsilon}$, defined, for the case of rigid - viscoplastic material models, by the relation

$$ {}^t\dot{\overline{\varepsilon}} = \frac{D\overline{\varepsilon}}{Dt} \tag{2} $$

with the notation $\frac{D(\bullet)}{Dt}$ we indicate time derivatives calculated following the particles, i.e. the *material time derivatives* [15].

For a rigid - viscoplastic solid, using Perzyna's constitutive model we write [19],

$$ {}^t s_{ij} = 2\mu({}^t\overline{\varepsilon}, {}^t\dot{\overline{\varepsilon}}) \, {}^t\dot{\varepsilon}_{ij} \tag{3} $$

For the $t-$equilibrium configuration, using the Principle of Virtual Work we write,

$$ \int_{{}^tV} ({}^t s_{ij} \, \delta\dot{\varepsilon}'_{ij} + {}^t p \, \delta\dot{\varepsilon}_v) \, {}^t dv = \int_{{}^tV} {}^t f_i^v \, \delta\dot{u}_i \, {}^t dv + \int_{{}^tS} {}^t t_i \, \delta\dot{u}_i \, {}^t ds \tag{4} $$

where,

${}^t\dot{\varepsilon}'_{ij}$	: Cartesian components of the deviatoric strain rate tensor;
${}^t p$	: hydrostatic component of the Cauchy stress tensor;
${}^t\dot{\varepsilon}_v$	: hydrostatic (volumetric) component of the strain rate tensor;
${}^t f_i^v$	: Cartesian components of the external forces per unit volume;
${}^t t_i$	: Cartesian components of the external forces per unit surface;
${}^t V$	: volume of the $t-$configuration;
${}^t S$	: external surface of the $t-$configuration.

We now consider a mixed finite element formulation [20] and at any point inside an element, $\underline{\dot{u}}$ is the velocity vector and $\underline{\dot{\varepsilon}}'$ is the column array containing the components of the deviatoric strain rate tensor. In our finite element formulation we define the following interpolations,

$$ \underline{\dot{u}} = \underline{H}_u \underline{\dot{U}} \tag{5a} $$

$$ \underline{\dot{\varepsilon}}' = \underline{B}_D \underline{\dot{U}} \tag{5b} $$

$$ \dot{\varepsilon}_V = \underline{\widetilde{B}}_V \underline{\dot{U}} \tag{5c} $$

where $\dot{\underline{U}}$ is the vector of nodal velocities, $\underline{B}_D$ is obtained from the velocities interpolation and $\widetilde{\underline{B}}_V$ is constructed so as to avoid locking. Different element formulations that do not lock and do not introduce spurious rigid body modes have been proposed in the literature [2], [20].

Using a penalty technique to impose incompressibility we have,

$$p = \kappa \, \dot{\varepsilon}_V \tag{6}$$

where κ is a large number [20] (e.g. for an element "e" we use $\kappa^{(e)} = 10^4$ to $10^7 \mu^{(e)}_{\max}$ [7]).

Therefore, after some algebra we get,

$$(\underline{K}_D + \underline{K}_V)\dot{\underline{U}} = \underline{R} \tag{7}$$

$$\underline{K}_D = \int_{tV} \underline{B}_D^T \, 2\mu \, \underline{B}_D \, dv \tag{8}$$

$$\underline{K}_V = \int_{tV} \kappa \, \widetilde{\underline{B}}_V^T \, \widetilde{\underline{B}}_V \, dv \tag{9}$$

$\underline{R}$: nodal forces equivalent to the external loads acting in the spatial configuration.

When using in 2D analyses 4-node quadrilateral elements or in 3D analyses 8-node hexahedral elements, to avoid locking $\widetilde{\underline{B}}_V$ is constructed considering a constant volumetric strain rate interpolation and therefore from Eqn.(6) a constant pressure interpolation [2] [20] [6].

Taking into account that the viscosity is a function of the accumulated effective viscoplastic strain ($\overline{\varepsilon}$) and of the effective strain rate ($\dot{\overline{\varepsilon}}$), it is obvious that Eqn. (7) represents a set of nonlinear equations that has to be solved using an iterative technique like the Newton - Raphson method [2]; hence, for the k-th iteration we write:

$$(\underline{K}_D + \underline{K}_V)^{(k-1)} \, \Delta\dot{\underline{U}}^{(k)} = \underline{R} - \underline{F}^{(k-1)} \tag{10a}$$

$$\dot{\underline{U}}^{(k)} = \dot{\underline{U}}^{(k-1)} + \Delta\dot{\underline{U}}^{(k)} \, . \tag{10b}$$

In the usual case, in which velocity boundary conditions are prescribed at some nodes, we proceed in the standard way (see Section 4.2.2 in [2]).

In Eqn. (10a) the matrices $\underline{K}_D^{(k-1)}$ and $\underline{K}_V^{(k-1)}$ are function of $\mu(\overline{\varepsilon}^{(k-1)}, \dot{\overline{\varepsilon}}^{(k-1)})$ and,

$$\underline{F}^{(k-1)} = \left[\int_{^tV} \underline{B}_D^T \, 2\mu^{(k-1)} \, \underline{B}_D \, dv \right] \, \overset{.}{\underline{U}}^{(k-1)} + \int_{^tV} \widetilde{\underline{B}}_V^T \, \overline{p} \, dv \; . \tag{11}$$

When the standard penalty method is used $\overline{p} = \kappa^{(k-1)} \, \widetilde{\underline{B}}_V \, \overset{.}{\underline{U}}^{(k-1)}$ and when the augmented Lagrangian method is used $\overline{p}$ is built in an iterative way [7].

The algebraic system in Eqn. (10a) presents two principal sources of bad conditioning:

- The eigenvalues related to the volume change modes are much larger than the eigenvalues related to the constant volume modes; even when using an augmented Lagrangian procedure, where κ can be smaller than in the standard penalty procedure.
- The viscosity, in Eqn. (3), is a strong function of the strain rate and therefore will present large variations between different points in the same configuration.

Also, when using the pseudo-concentrations technique, [17] and [18], we consider a fixed mesh with the material moving inside it. For those mesh points in which no material is present we adopt a reduced viscosity value and therefore, this is also a source of bad conditioning (see Section 6).

3 The Conjugate Gradient Method

The Conjugate Gradient Method (CG) [1], [14] is an effective algorithm for solving, in an iterative way, symmetric positive definite systems, such as the one in Eqn. (10a).

Given the linear system,

$$\underline{A} \, \underline{x} = \underline{b} \tag{12}$$

the convergence rate of the CG method depends on spectral properties of the coefficients matrix $\underline{A}$.

Calling $\lambda_{\max}$ the maximum eigenvalue of $\underline{A}$, and $\lambda_{\min}$ its minimum eigenvalue, the spectral condition number for the symmetric positive define matrix is $\kappa_2 = (\lambda_{\max}/\lambda_{\min})$ [5].

If $\underline{x}_s$ is the exact solution of Eqn. (12), then it can be shown [14] that in the CG method for the $j-th$ iteration,

$$\left\| \underline{x}^{(j)} - \underline{x}_s \right\|_A \leq 2 \, \rho^j \, \left\| \underline{x}^{(0)} - \underline{x}_s \right\|_A \tag{13a}$$

$$\rho = \frac{\sqrt{\kappa_2} - 1}{\sqrt{\kappa_2} + 1} \tag{13b}$$

where $\|\underline{v}\|_A \equiv \underline{v}^T \underline{A} \underline{v}$.

From Eqns. (13a) and (13b) we see that for $\kappa_2 = 1$ the convergence of the CG method is optimum and for $\kappa_2 \longrightarrow \infty$ the method does not converge.

Hence, one may attempt to transform the linear system (12) into one that is equivalent in the sense that it has the same solution *but* presents more favorable spectral properties. We use a preconditioning matrix $\underline{P}$, and the equations system is transformed into,

$$\underline{P}^T \underline{A} \underline{P} \, (\underline{P}^{-1} \underline{x}) = \underline{P}^T \underline{b} \, . \tag{14}$$

The goal of the preconditioner is to reduce the spectral number of the coefficients matrix.

4 The Canga - Becker preconditioner

In this section we discuss our implementation of the preconditioner developed in Ref. [3] by Canga and Becker for solving, via the CG method, the equation system in (10a); we specialize our discussion for the 8-node hexahedral element with constant pressure and volumetric strain rate interpolations (*H1-P0* element) [7].

4.1 Spectral analysis

For an element with a constant penalty parameter and volume $V^{(e)}$ we have,

$$\underline{K}_V^{(e)} = \kappa \int_{V^{(e)}} \widetilde{\underline{B}}_V^T \, \widetilde{\underline{B}}_V \, dv = \kappa \widehat{\underline{K}}_V \, . \tag{15}$$

For the element formulation we are considering (*H1-P0*) $\widetilde{\underline{B}}_V$ is constant inside each element.

We now define the vectors $\underline{b}^{(e)}$ $e = 1, ..., NEL$ where NEL is the number of elements in the model:

$$\underline{b}^{(e)} = \widetilde{\underline{B}}_V^T \, \sqrt{V_e} \, . \tag{16}$$

Hence, for the (*H1-P0*) element,

$$\widehat{K}_V = \sum_{j=1}^{NEL} \underline{b}_j \otimes \underline{b}_j^T \tag{17}$$

where we call $\underline{b}_j$ the element vectors $\underline{b}^{(e)}$ extended to the complete domain $\mathcal{R}^{NEQ}$. From the above we see that $\underline{K}_V$ has a rank NEL since the $\underline{b}_j$ are linearly independent vectors.

4.2 Velocity boundary conditions

If the velocity boundary conditions are not constant all along the system evolution (e.g. contact conditions), it will be necessary to include the corresponding degrees of freedom in the problem resolution. This is because the number of the nonzero coefficients of the stiffness matrix, storaged in a sparse format, is calculated just once. Hence, we impose:

$$(K)_{i,j} = 0 \qquad \forall i \neq j, \ for \ \underset{\cdot}{\underline{U}}^{(i)} \ fixed \tag{18}$$

where $\underline{K}$ is the global stiffness matrix.

It is known that the bulk modes of deformation separate from the set corresponding to the distortional modes of deformation at a rate proportional to the bulk modulus κ; but within each set, the spread of eigenvalues remains bounded.

The modes corresponding to the degrees of freedom with fixed velocity, will also separate form the distortional modes at a rate proportional to κ. That's easy to deduce since the eigenvalue corresponding to the degree of freedom "v" will be $K_{vv} \cong cte^{*}\kappa$.

Considering $NVELO$ prescribed d.o.f. we can define:

For $j = 1, ..., NEL$

- $\underline{c}_j(i) = \underline{b}_j(i)$ if the corresponding velocity d.o.f., "i", is not fixed.
- $\underline{c}_j(i) = 0$ if the corresponding velocity d.o.f., "i", is fixed.

For $j = (NEL + 1), ..., (NEL + NVELO)$

- $\underline{c}_j(i) = 0$ if the corresponding velocity d.o.f., "i", is not fixed.
- $\underline{c}_j(i) = 1$ if the corresponding velocity d.o.f., "i", is fixed.

Therefore,

$$\underline{K} = \underline{K}_D \ + \ \kappa \sum_{j=1}^{(NEL+NVELO)} \underline{c}_j \otimes \underline{c}_j^T \tag{19}$$

4.3 The preconditioner

In this subsection we discuss a preconditioning scheme suitable for improving the conditioning of the above described stiffness matrix and therefore suitable for improving the convergence ratio when solving the system of equations using the (CG) method.

In a problem with NEQ equations we can define a subspace $\mathcal{B}$ in $\mathcal{R}^{NEQx(NEL+NVELO)}$ spanned by the $(NEL + NVELO)$ orthogonal vectors $\{\underline{c}_j\}$ of dimension NEQ that we defined in the previous subsection. The projection operator on $\mathcal{B}$, considering that the base vectors are normalized, is

$$\underline{P}_B = \left[\underline{c}_1\ \underline{c}_2\ \cdots\ \underline{c}_{(NEL+NVELO)}\right] \begin{bmatrix} \underline{c}_1^T \\ \underline{c}_2^T \\ \cdots \\ \cdots \\ \underline{c}_{(NEL+NVELO)}^T \end{bmatrix} = \sum_{j=1}^{(NEL+NVELO)} \underline{c}_j \otimes \underline{c}_j^T \tag{20}$$

and $\underline{P}_B\ \epsilon\ \mathcal{R}^{NEQxNEQ}$.

It is straightforward to show that $\underline{P}_B = \underline{P}_B^T$ and that $(\underline{P}_B)^2 = \underline{P}_B$; therefore $\underline{P}_B(\mathcal{I} - \underline{P}_B) = 0$.

The projection operator that projects on the orthogonal complement of $\mathcal{B}$ is,

$$\begin{aligned} \underline{P}_B^{\perp} &= \underline{I} - \left[\underline{c}_1\ \underline{c}_2\ \cdots\ \underline{c}_{(NEL+NVELO)}\right] \begin{bmatrix} \underline{c}_1^T \\ \underline{c}_2^T \\ \cdots \\ \cdots \\ \underline{c}_{(NEL+NVELO)}^T \end{bmatrix} \\ &= \underline{I} - \sum_{j=1}^{(NEL+NVELO)} \underline{c}_j \otimes \underline{c}_j^T \end{aligned} \tag{21}$$

In the constant viscosity case (Newtonian fluid) the stiffness matrix in Eqn. (19) can be written as,

$$\underline{K}_{(\mu=const)} = \mu\ \widehat{\underline{K}}_D + \kappa \sum_{j=1}^{(NEL+NVELO)} \underline{c}_j \otimes \underline{c}_j^T\ . \tag{22}$$

Canga and Becker introduced the following projector operator [3],

$$\underline{P} = \omega \underline{P}_B + \underline{P}_B^{\perp} \tag{23}$$

where $\omega = \sqrt{\mu\ /\ \kappa}$. Applying the above projector operator to the matrix in Eqn. (22) the resulting preconditioned matrix reduces to an expression totally independent of bulk modulus when $\kappa >> \mu$

If the vectors $\underline{c}_j$ that span $\mathcal{B}$ are not orthonormal; then,

$$\begin{aligned} \underline{P}_B &= \underline{B} \begin{bmatrix} \beta_{11} & \cdots & \beta_{1(NEL+NVELO)} \\ \beta_{21} & \cdots & \beta_{2(NEL+NVELO)} \\ \cdots & \cdots & \cdots \\ \beta_{(NEL+NVELO)1} & \cdots & \beta_{(NEL+NVELO)(NEL+NVELO)} \end{bmatrix} \underline{B}^T \\ &= \sum_{i,Pj=1}^{(NEL+NVELO)} \beta_{ij}\ \underline{c}_i \otimes \underline{c}_j^T\ . \end{aligned} \tag{24}$$

In the above we defined,

$$\underline{B} = \left[\underline{c}_1\ \underline{c}_2\ ...\ \underline{c}_{(NEL+NVELO)}\right] \tag{25a}$$

$$\underline{B}_{norm} = \underline{B}^T\underline{B}\,. \tag{25b}$$

Hence,

$$\beta_{ij} = (\underline{B}^{-1}_{norm})_{ij} \tag{26}$$

and replacing in Eqn. (23) we get,

$$\underline{P} = \underline{I} + (\omega - 1)\ \underline{B}\underline{B}^{-1}_{norm}\underline{B}^T\ . \tag{27}$$

Let the vectors that span the orthogonal complement of $\mathcal{B}$: $\mathcal{B}^\perp$ be $\{\underline{v}_i\}$; $i = 1, ..., NEQ - NEL - NVELO$.

4.3.1 Newtonian fluids

For a constant viscosity body the preconditioner takes the form:

$$\underline{P} = \omega \sum_{i,j=1}^{(NEL+NVELO)} \beta_{ij}\ \underline{c}_i \otimes \underline{c}_j^T\ + \sum_{i,j=1}^{NEQ-NEL-NVELO} \gamma_{ij}\ \underline{v}_i \otimes \underline{v}_j^T \tag{28}$$

where, in the same way as before we defined γ_{ij} as the components of the inverse of the norm matrix in the space $\mathcal{B}^\perp$.

Pre and post-multiplying the stiffness matrix in Eqn. 22 we get,

$$\begin{aligned}
\underline{P}^T\underline{K}\underline{P} = {} & \omega^2\,\mu \sum_{i,k,s,t=1}^{(NEL+NVELO)} \beta_{ik}\,\beta_{st}\ \left(\widehat{\underline{K}}_D\right)_{ks}\ \underline{c}_i \otimes \underline{c}_t^T \\
& +\omega\,\mu \sum_{i,k=1}^{(NEL+NVELO)} \sum_{p,z=1}^{(NEQ-NEL-NVELO)} \beta_{ik}\,\gamma_{pz}\ \left(\widehat{\underline{K}}_D\right)_{kp}\ \underline{c}_i \otimes \underline{v}_z^T \\
& +\omega\mu \sum_{m,l=1}^{(NEQ-NEL-NVELO)} \sum_{s,t=1}^{(NEL+NVELO)} \gamma_{ml}\,\beta_{st}\ \left(\widehat{\underline{K}}_D\right)_{ls}\ \underline{v}_m \otimes \underline{c}_t^T \\
& +\omega^2\kappa \sum_{i,k,j,s,t=1}^{(NEL+NVELO)} \beta_{ik}\,\beta_{st}\ \left(\underline{c}_k \bullet \underline{c}_j\right)\left(\underline{c}_j \bullet \underline{c}_s\right)\ \underline{c}_i \otimes \underline{c}_t^T \\
& +\mu \sum_{m,l,p,z=1}^{(NEQ-NEL-NVELO)} \gamma_{ml}\,\gamma_{pz}\ \left(\widehat{\underline{K}}_D\right)_{lp}\ \underline{v}_m \otimes \underline{v}_z^T
\end{aligned} \tag{29}$$

where we called $\left(\underline{\widehat{K}}_D\right)_{ab}$ the matrix components in the base of $\mathcal{R}^{NEQxNEQ}$ $\left\{\underline{c}_j, \underline{v}_i\right\}$. When $\kappa = 10^\alpha\ \mu$ and $\kappa \longrightarrow \infty$ we are left with,

$$\begin{aligned}\underline{P}^T\underline{K}\underline{P} &= \mu\,(\sum_{i,k,j,st=1}^{(NEL+NVELO)} \beta_{ik}\,\beta_{st}\ \left(\underline{c}_k \bullet \underline{c}_j\right)\left(\underline{c}_j \bullet \underline{c}_s\right)\ \underline{c}_i \otimes \underline{c}_t^T \\ &+ \sum_{m,l,p,z=1}^{(NEQ-NEL-NVELO)} \gamma_{ml}\,\gamma_{pz}\ \left(\underline{\widehat{K}}_D\right)_{lp}\ \underline{v}_m \otimes \underline{v}_z^T) \end{aligned} \tag{30}$$

and the condition number of the matrix $\underline{P}^T\underline{K}\underline{P}$ is independent of the bulk modulus, κ.

In order to display the effect of the above discussed preconditioner on the conditioning of the stiffness matrix, in Table 1 we compare the preconditioned matrices of two hexahedral elements with very different aspect ratios (a, b, c are the side lengths of the elements)

Hexahedral element dimensions	$a = b = c$	$100a = b = 10c$
Cond $(\underline{K})$ $(\frac{\max\ eigenvalue}{\min\ eigenvalue})$	$4.2 \cdot 10^7$	$2.4 \cdot 10^{10}$
Cond $(\underline{P}^T\underline{K}\underline{P})$ $(\frac{\max\ eigenvalue}{\min\ eigenvalue})$	6.3	$4,329$

Table 1. Elements with different aspect ratios and constant viscosity $\left(\kappa = \mu\ 10^7\right)$

We see from the above results that the discussed preconditioner eliminates the bad conditioning coming from the penalty parameter *but* does not alleviate the condition number imposed by the aspect ratio.

4.3.2 Non-Newtonian fluids

For non-Newtonian fluids or rigid-viscoplastic materials, the viscosity is not constant, it is a function of the accumulated effective plastic strain and effective strain rate; hence, it takes different values inside an element and between elements. In the rigid - viscoplastic material model that we consider for simulating metal forming operations, we take for each element the penalty coefficient $\kappa^{(e)} = 10^\alpha\ \nu^{(e)}$ where $\nu^{(e)}$ is an element parameter; hence, the parameter ω will have to change from element to element since we want to scale independently each of the components projected on the subspace $\mathcal{B} = \left\{\underline{c}_j\right\};\ j = 1, ..., (NEL + NVELO)$.

The projection of a vector $\underline{x}\ \epsilon\ \mathcal{R}^{NEQ}$ on $\mathcal{B}$ can be expressed as,

$$\underline{\widehat{x}} = \sum_{j=1}^{(NEL+NVELO)} \alpha_j\ \underline{c}_j = \underline{B}\underline{\alpha} \tag{31a}$$

$$\underline{\alpha} = \left(\underline{B}^T\underline{B}\right)^{-1}\underline{B}^T\underline{x} \tag{31b}$$

We now want to find an operator, $\underline{P}^{\omega}$ that when applied to a vector in $\mathcal{R}^{NEQ}$ produces a vector in $\mathcal{B}$ but with its components scaled with different parameters ω_j. Therefore, for the case of variable viscosity, we re-write the preconditioner in Eqn. (28) as,

$$\underline{P} = \sum_{i,j=1}^{(NEL+NVELO)} \omega_i \, \beta_{ij} \, \underline{c}_i \otimes \underline{c}_j^T + \sum_{i,j=1}^{NEQ-NEL-NVELO} \gamma_{ij} \, \underline{v}_i \otimes \underline{v}_j^T \qquad (32)$$

Even tough this preconditioner is not symmetric, the product $\underline{P}^T \underline{K} \underline{P}$ is still symmetric and therefore we can use the CG method to solve the linear system of equations.

The preconditioned stiffness matrix is now,

$$\begin{aligned}
\underline{P}^T \underline{K} \underline{P} \;=\; & \sum_{i,k,s,t=1}^{(NEL+NVELO)} \omega_i \, \omega_s \, \beta_{ik} \, \beta_{st} \, \left(\underline{K}_D\right)_{ks} \, \underline{c}_i \otimes \underline{c}_t^T \\
& + \sum_{i,k=1}^{(NEL+NVELO)} \sum_{p,z=1}^{(NEQ-NEL-NVELO)} \omega_i \, \beta_{ik} \, \gamma_{pz} \, \left(\underline{K}_D\right)_{kp} \, \underline{c}_i \otimes \underline{v}_z^T \\
& + \sum_{m,l=1}^{(NEQ-NEL-NVELO)} \sum_{s,t=1}^{(NEL+NVELO)} \omega_s \, \gamma_{ml} \, \beta_{st} \, \left(\underline{K}_D\right)_{ls} \, \underline{v}_m \otimes \underline{c}_t^T \\
& + \sum_{i,k,j,st=1}^{(NEL+NVELO)} \omega_i \, \omega_s \, \beta_{ik} \, \beta_{st} \, \kappa_j \, \left(\underline{c}_k \bullet \underline{c}_j\right) \left(\underline{c}_j \bullet \underline{c}_s\right) \, \underline{c}_i \otimes \underline{c}_t^T \\
& + \sum_{m,l,p,z=1}^{(NEQ-NEL-NVELO)} \gamma_{ml} \, \gamma_{pz} \, \left(\underline{K}_D\right)_{lp} \, \underline{v}_m \otimes \underline{v}_z^T \qquad (33)
\end{aligned}$$

Let us consider, in a first approach:

- A constant viscosity inside each element but, different from element to element (μ_e).
- A penalty parameter $\kappa_e = 10^{\alpha} \, \nu_e$, where ν_e is an element parameter.

For this case we propose to use in the preconditioner,

$$\omega_e = \sqrt{\frac{\mu_e}{10^{\alpha} \, \nu_e}} \quad e = 1, ..., NEL. \qquad (34)$$

For the d.o.f. with prescribed velocities we use $\mu_i = \nu_i = 1$.
When $10^{\alpha} \longrightarrow \infty$ the Eqn. (33) reduces to,

$$\underset{\kappa\to\infty}{\underline{P}^T\underline{K}\underline{P}} = \sum_{i,k,s,t=1}^{(NEL+NVELO)} \sqrt{\frac{\mu_i}{\nu_i}}\sqrt{\frac{\mu_s}{\nu_s}}\,\beta_{ik}\,\beta_{st} \left[\sum_{e=1}^{(NEL+NVELO)} \nu_e\ (\underline{c}_k \bullet \underline{c}_e)\,(\underline{c}_e \bullet \underline{c}_s)\right] \underline{c}_i \otimes \underline{c}_t^T + \sum_{m,l,p,z=1}^{(NEQ-NEL-NVELO)} \gamma_{ml}\,\gamma_{pz} \left[\sum_{e=1}^{NEL} \mu_e\ \left(\widehat{\underline{K}}_D\right)^e_{lp}\right] \underline{v}_m \otimes \underline{v}_z^T \tag{35}$$

We can analyze the following specific cases:

- For cubic elements the vectors $\left\{\underline{c}_j\right\}$ and $\left\{\underline{v}_j\right\}$ are orthogonal, when we normalize them, we get,

$$\underset{\kappa\to\infty}{\underline{P}^T\underline{K}\underline{P}} = \sum_{e=1}^{(NEL+NVELO)} \mu_e \left[\underline{c}_e \otimes \underline{c}_e^T + \sum_{m,p=1}^{(NEQ-NEL-NVELO)} \left(\widehat{\underline{K}}_D\right)^e_{mp} \underline{v}_m \otimes \underline{v}_p^T\right] + \sum_{e=(NEL+1)}^{(NEL+NVELO)} \mu_e \underline{c}_e \otimes \underline{c}_e^T \ . \tag{36}$$

For this case the matrix $\underline{P}^T\underline{K}\underline{P}$ is independent of the penalty parameter.

- For the case in which inside each element $\mu_e = \nu_e$,

$$\underset{\kappa\to\infty}{\underline{P}^T\underline{K}\underline{P}} = \sum_{i,k,s,t=1}^{(NEL+NVELO)} \beta_{ik}\,\beta_{st} \left[\sum_{e=1}^{(NEL+NVELO)} \mu_e\ (\underline{c}_k \bullet \underline{c}_e)\,(\underline{c}_e \bullet \underline{c}_s)\right] \underline{c}_i \otimes \underline{c}_t^T + \sum_{m,l,p,z=1}^{(NEQ-NEL-NVELO)} \gamma_{ml}\,\gamma_{pz} \left[\sum_{e=1}^{NEL} \mu_e\ \left(\widehat{\underline{K}}_D\right)^e_{mp}\right] \underline{v}_l \otimes \underline{v}_z^T \tag{37}$$

The spread between the eigenvalues of $\underline{P}^T\underline{KP}$ is determined by the differences between the element viscosities.

- For the general case ($\mu_e \neq \nu_e$) our numerical experience has shown us that the most convenient choice is,

$$\omega_e = \sqrt{\frac{1}{10^\alpha}} \quad e = 1, ..., NEL \tag{38}$$

therefore, when $10^\alpha \longrightarrow \infty$ the preconditioned matrix takes the form,

$$\begin{aligned} \underline{P}^T\underline{KP} \quad = \quad & \sum_{i,k,s,t=1}^{(NEL+NVELO)} \beta_{ik}\,\beta_{st} \\ & \left[\sum_{e=1}^{(NEL+NVELO)} \nu_e\,(\underline{c}_k \bullet \underline{c}_e)\,(\underline{c}_e \bullet \underline{c}_s)\right] \underline{c}_i \otimes \underline{c}_t^T \\ & + \sum_{m,l,p,z=1}^{(NEQ-NEL-NVELO)} \gamma_{ml}\,\gamma_{pz} \\ & \left[\sum_{e=1}^{NEL} \mu_e \left(\widehat{\underline{K}}_D\right)^e_{lp}\right] \underline{v}_m \otimes \underline{v}_z^T \end{aligned} \tag{39}$$

Equation (38) is therefore used to adapt the Canga - Becker preconditioner to the case of variable viscosity and when the penalty parameter is a function of the elements viscosity.

5 The Jacobi preconditioner

In the second section of this paper, we discussed the flow formulation and the two main sources of bad conditioning for the algebraic system of equations that has to be solved at every iteration step. In the previous section we discussed a preconditioning scheme aimed at removing the bad conditioning that comes from the penalty imposition of the incompressibility constraint.

For the bad conditioning induced by the large differences in the viscosity values among different points inside the model we implemented the Jacobi preconditioner,

$$(J)_{ij} = \left(\sum_{e=1}^{NEL} \mu_e^{\max} \widehat{K}_D^e\right)^{-1/2}_{ii} \delta_{ij}\ . \tag{40}$$

We use both pre-conditioners in cascade: first the Jacobi preconditioner and afterwards, on the Jacobi-preconditioned stiffness matrix, we use the Canga - Becker preconditioner. In Table 2 we present a very simple example showing the effect of both preconditioners acting in cascade.

Hexahedral element dimensions	$a = b = c$
Cond $(\underline{K})$ $(\frac{\max\ eigenvalue}{\min\ eigenvalue})$	$8.1 \cdot 10^{11}$
Cond $(\underline{P}^T \underline{K} \underline{P})$ $(\frac{\max\ eigenvalue}{\min\ eigenvalue})$	$8.1 \cdot 10^{3}$
Cond $(\underline{P}^T \underline{J} \underline{K} \underline{J} \underline{P})$	4.61

Table 2. Two elements Stokes problem ($\mu_1 = 0.1$; $\mu_2 = 2000$; $\kappa = \mu\ 10^7$)

It is interesting to note that the Jacobi preconditioner in Eqn.(40) is not useful for overcoming the ill conditioning due to a bad aspect ratio, in the next table we present an example that confirms this assertion,

Hexahedral element dimensions	$a = b = c$	$100\ a = b = 10\ c$
Cond $(\underline{P}^T \underline{K} \underline{P})$ $(\frac{\max\ eigenvalue}{\min\ eigenvalue})$	6.3	4,329.
Cond $(\underline{P}^T \underline{J} \underline{K} \underline{J} \underline{P})$	4.0	4,330.

Table 3. Stokes problem ($\mu = const$; $\kappa = \mu\ 10^7$)

6 Numerical experimentation

In this section we discuss several examples that illustrate on the performance of the above discussed iterative solving scheme.

The examples are solved using our finite element code **METFOR** ([4], [7] to [12]) which for solving rigid - viscoplastic models uses the flow formulation [19] implemented via the pseudo - concentrations technique [17] [18]. The 3D examples discussed in what follows were modeled using the (*H1-P0*) element and the incompressibility constraint was imposed using the augmented Lagrangian procedure.

The finite element models are solved using the following staggered procedure:

- Variables interpolated from their nodal values,

$\dot{\underline{u}}$: material velocities,
c : nodal pseudo-concentrations

$c \geq 0$ actual material is present at the point,
$c < 0$ there is no material present at the point
(a small μ is considered to avoid singular stiffness matrices [12])

- The mesh is fixed and, knowing the velocity field we can calculate the c-distribution with the following transport equations:

$$\dot{\underline{u}} \cdot \underline{\nabla} c = 0 \ (\textit{stationary problems}) \tag{41a}$$

$$\frac{\partial c}{\partial t} + \dot{\underline{u}} \cdot \underline{\nabla} c = 0 \ (\textit{transient problems}) \tag{41b}$$

- The equivalent plastic strains are transported using,

$$\dot{\underline{u}} \cdot \underline{\nabla} \overline{\varepsilon} = \frac{\langle c \rangle}{|c|} \dot{\overline{\varepsilon}} \ (\textit{stationary problems}) \tag{42a}$$

$$\frac{\partial \varepsilon}{\partial t} + \dot{\underline{u}} \cdot \underline{\nabla} \overline{\varepsilon} = \frac{\langle c \rangle}{|c|} \dot{\overline{\varepsilon}} \ (\textit{transient problems}) \tag{42b}$$

where $\langle \cdot \rangle$ is the Macauley bracket.

- For stationary problems we start the iterative algorithm from a trial c-distribution and zero trial velocities: $\dot{\underline{u}} = \underline{0}$.
- For transient problems ($t \longrightarrow t + \Delta t$) we start the iterative algorithm from the converged solution of the previous step.

1. $l = -1$
2. $l = l + 1$
   - (i) $j = 0$; $\dot{\underline{u}}^{(j)} = \dot{\underline{u}}^{(j-1)}$
   - (ii) $j = j + 1$
     Solve the work-piece nonlinear equilibrium equations keeping constant the c-distribution and the $\overline{\varepsilon}$-distribution
     $\dot{\underline{u}}^{(j)} = f\,(\,\dot{\underline{u}}^{(j-1)},\ c^{(l)},\ \overline{\varepsilon}^{(l)})$
   - (iii) $IF\ \dfrac{\left\| \dot{\underline{u}}^{(j)} - \dot{\underline{u}}^{(j-1)} \right\|_2}{\left\| \dot{\underline{u}}^{(j)} \right\|_2} \leq UTOL\ .AND.\ \left\| \dot{\varepsilon}_v \right\|_\infty \leq VTOL$
     $THEN \longrightarrow \dot{\underline{u}}^{(l)} = \dot{\underline{u}}^{(j)}\ GO\ TO\ 3$
     $ELSE \longrightarrow\ GO\ TO$ 2.(ii)
3. *Calculate the c-distribution and $\overline{\varepsilon}$-distribution solving the corresponding transport equations*
4. $IF\ l = 0\ GO\ TO\ 2$
   $ELSE \longrightarrow IF \dfrac{\left\| \dot{\underline{u}}^{(l)} - \dot{\underline{u}}^{(l-1)} \right\|_2}{\left\| \dot{\underline{u}}^{(l)} \right\|_2} \leq UTOL$
   $THEN \longrightarrow CONVERGENCE$
   $ELSE \longrightarrow GO\ TO\ 2$

Box I: Staggered iterative algorithm for coupling the equilibrium equations to the $c-$transport equations

During the step (2.ii) the penalty parameter is not kept constant as in [7], instead it is changed at every iteration so as to have $\kappa = 10^{\alpha}\ \mu_{\max}^{elem}$; where

$\mu_{\max}^{elem}$ is the maximum element viscosity obtained from the velocity distribution produced by the previous iteration. In the following numerical experimentation we use $\alpha = 4$.

Skyline storage is used for the direct solver (*COLSOL* in [2]) and compressed row storage for the iterative solvers. The performance of the direct and iterative solvers is compared in terms of CPU time and storage requirements. The convergence tolerance considered for the iterative solvers at each iteration was taken as:

$$\frac{\left\|\underline{r}^i\right\|}{\left\|\underline{r}^0\right\|} \leq 10^{-8}$$

where $\underline{r}^i$ is the residual at the i-th solver iteration.

For calculating the storage required by the direct solver we consider the coefficient matrix alone whereas for the iterative solvers it includes the compacted coefficient matrix, the vectors of indices and the preconditioner storage.

All problems were run in a workstation using double precision.

6.1 Extrusion of a square bar

In Fig. 1 we present the finite element model that we used for solving this problem under the assumption that the material completely fills the extrusion die; since there are no free surfaces, $c > 0$ everywhere and there is no need for solving the c-transport equations.

In what follows we present the comparison between the performance of the direct solver and the iterative solver (*CG + Canga - Becker preconditioner + Jacobi preconditioner*) considering two models:

6.1.1 Model 1

Number of equations :$3,311$
Number of elements: $10x10x10$

	Storage [Mb]	**Relative CPU time**	**Iterations** [1]
Direct Solver	8.7	1.00	–
Iterative Solver	2.4	0.73	1993

Table 4. Performance of the iterative solver for the smallest model of the square bar extrusion

[1] Total number of iterations used by the iterative solver to solve all the linear systems in the iterative N-R scheme used to determine the velocity field.

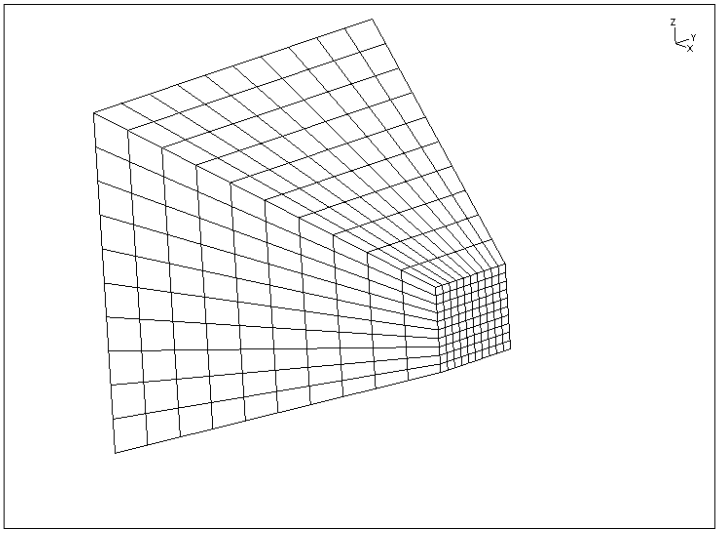

Figure 1: Finite element model for simulating the extrusion of a square bar (rigid / perfectly-plastic material model)

6.1.2 Model 2

Number of equations : $8,835$
Number of elements: $14x14x14$

	Storage [Mb]	**Relative CPU time**	**Iterations**
Direct Solver	44.4	1.00	–
Iterative Solver	8.9	0.49	4753

Table 5. Performance of the iterative solver for the largest model of the square bar extrusion

6.2 Swelling of a non-Newtonian planar flow

In this problem we consider the swelling of a planar flow flowing through a channel, when it reaches an abrupt expansion; a non-Newtonian fluid was considered with

$$\mu = \frac{1}{3}\left(\dot{\bar{\varepsilon}}\right)^{-0.5}$$

The problem was solved using the pseudo-concentrations technique as in [12] but in this case 3D elements were used.

For the proposed model we have $8,415$ d.o.f. and the performance of different solvers is compared in Table 6. For a description of the different iterative schemes tested please refer to [1], [14].

Solver	**Storage (Mb)**	**Iterations per N-R it.**	**Relative CPU time**
DIRECT	28.6	–	1.0
GMRES + Jac	6.9		315.
CG + Jac	3.6	7800	8.8
CG + SSOR	3.7	3235	9.5
CG + ICHOL (no fill in)	7.0	1775	10.1
CG + ICHOL (with fill in)	7.2	1796	15.1
CG + C-B + Jac	6.8	315	0.8

Notes:

1. *For the GMRES solver the dimension of the Krylov space was set as 50 [1].*

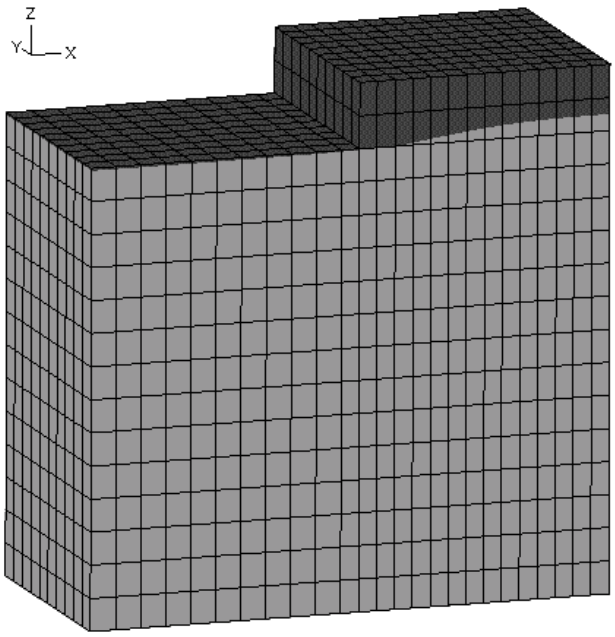

Figure 2: Swelling of a Newtonian fluid

2. *Jac: Jacobi preconditioner.*

3. *SSOR: preconditioning with symmetric succesive over-relaxation [1]*

4. *ICHOL:preconditioning with incomplete Cholevsky factorization [1].*

5. *C-B: the Canga - Becker preconditioner.*

Table 6. Performance of the different iterative solvers

From the results in Table 6 it is clear that for this problem the (CG) method coupled to the Canga - Becker and Jacobi preconditioners presents the best performance.

6.3 Piercing

In Fig. 3 we present an exploded graph of a finite element model that simulates the piercing process of a rigid - perfectly plastic bar (Mannesmann process). The model was developed using the flow formulation implemented via the pseudo-concentrations technique, and in the figure we show:

- Finite element mesh with the pseudo-concentrations distribution (the dark zones indicate $c > 0$).
- Rolling tools: rolls, guiding shoes and plug that impose the actual boundary conditions on the material.

For this case we use a larger penalty coefficient in the augmented Lagrangian procedure that imposes the incompressibility constraint: $\kappa^{(e)} = 10^5 \ \mu^{(e)}_{\max}$.

The model has $67,076$ d.o.f. and in Table 7 we compare the average results that we obtain for each Newton - Raphson iteration using either the direct or the iterative solver (*CG + C-B + Jac*).

Solver	**Storage (Mb)**	**Relative CPU time**
DIRECT	$1,286$	1.00
CG + C-B + Jac	0.15	0.81

Table 7. Piercing of a rigid - perfectly plastic bar

It is important to remark that in the above finite element model we have included elements with very poor aspect ratios.

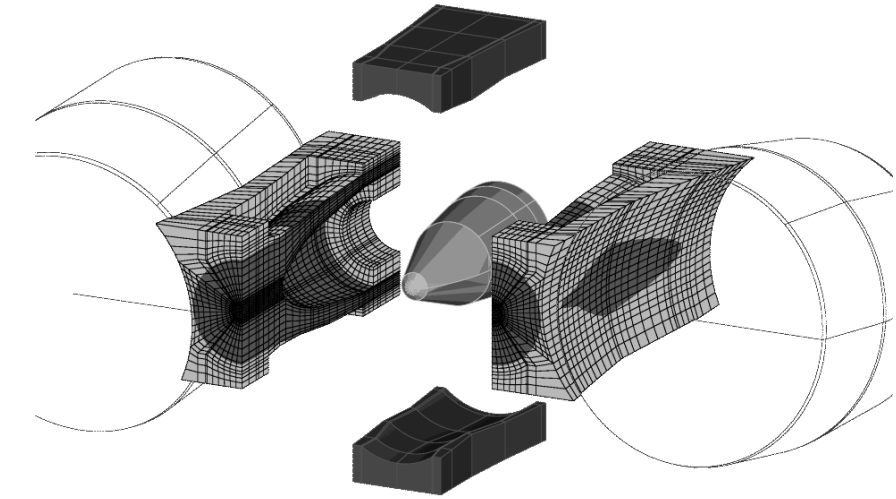

Figure 3: Finite element mesh for modeling the piercing of a rigid - perfecly plastic bar

7 Conclusions

A preconditioned (C-G) iterative solver was implemented to solve the bad conditioned linear equation systems that have to be solved when modeling metal forming problems using rigid - viscoplastic material models (flow formulation).

An ad hoc preconditioning scheme was analyzed and implemented considering the two main sources of bad conditioning:

1. The stiffness spread between the volumetric and distortional deformation modes. For this purposes the Canga - Becker preconditioner was adapted for the case of the non-homogeneous mechanical properties that in a rigid - viscoplastic material models are induced by the non-homogeneous strain and strain rate distributions.

2. The stiffness spread induced by the large differences in the viscosity values among different points inside the model (this differences are enhanced when using the pseudo concentrations technique); for this purpose a Jacobi preconditioner was implemented acting in cascade with the Canga - Becker preconditioner.

The obtained iterative solver is very efficient considering both, the required storage and the processing CPU time, even tough the bad conditioning induced by the elements aspect ratio has not been tackled yet.

Acknowledgement 1 *The finacial support of the TECHINT steel companies is gratefully acknowledged.*

References

[1] Barrett R., Berry M., Chan T.F., Demmel J., Donato J.M., Dongarra J., Eijkhout V., Pozo R., Romine C. and Van de Vost H., *Templates for the Solution of Linear Systems: Building Blocks for iterative Methods,* SIAM Publications, 1994.

[2] Bathe K.J., *Finite Element Procedures*, Prentice Hall, 1996.

[3] Canga M.E. and Becker E.B., "An iterative technique for the finite element analysis of near-incompressible materials", *Comput. Methods Appl. Mech. Engrg.*, **170**, pp.79-101, 1999.

[4] Cavaliere M.A., Goldschmit M.B. and Dvorkin E.N., "3D modeling of bulk metal forming processes via the flow formulation and the pseudo - concentrations technique", *Proceedings Fifth Int. Conf. on Computational Plasticity*, (Ed. D.R.J.Owen et al), CIMNE, 1997.

[5] Conte S.D. and De Boor C., *Elementary Numerical Analysis*, Mc Graw-Hill, 1972.

[6] Dvorkin E.N., "On the convergence of incompressible finite element formulations: the Patch Test and the Inf-Sup condition", *Engng. Computations*, (in press)

[7] Dvorkin E.N., Cavaliere M.A. and Goldschmit M.B., "A three field element via augmented Lagrangian for modelling bulk metal forming processes", *Computational Mechanics*, **17**, pp.2-9, 1995.

[8] Dvorkin E.N., Cavaliere M.A., Goldschmit M.B. and Amenta P.M., "On the modeling of steel product rolling processes", *Int.J.Forming Processes (ESAFORM)*, **1**, pp.211-242, 1998.

[9] Dvorkin E.N., Cavaliere M.A., Zielonka M.G. and Goldschmit M.B., "New developments for the modeling of metal rolling processes", *Proceedings European Conference on Computational Mechanics* (Ed. W. Wunderlich et al), München-Germany, 1999.

[10] Dvorkin E.N., Goldschmit M.B., Cavaliere M.A. and Amenta P.M., "On the modelling of bulk metal forming processes", *Proc. Second ECCOMAS* (European Community on Computational Methods in Applied Sciences) Conference on Num. Methods in Engng., John Wiley & Sons, 1996.

[11] Dvorkin E.N., Goldschmit M.B., Cavaliere M.A., Amenta M.P., Marini O. and Stroppiana W., "2D finite element parametric studies of the flat rolling process", *J. of Materials Processing Technology*, **68**, pp.99-107, 1997.

[12] Dvorkin E.N. and Petöcz E.G., "An effective technique for modelling 2d metal forming processes using an Eulerian formulation", *Engng. Computations*, **10**, pp.323-336, 1993.

[13] Felippa C.A., "Iterative procedures for improving penalty function solutions of algebraic systems", *Int.J.Num.Meth.Engrg.,* **12**, pp.821-836, 1978.

[14] Golub G.H. and Van Loan C.F., *Matrix Computations*, The John Hopkins University Press, 1996.

[15] Malvern L.E. (1969), *Introduction to the Mechanics of a Continuous Medium,* Prentice Hall.

[16] Perzyna P., "Fundamental problems in viscoplasticity", *Advances in Applied Mechanics*, **9**, Academic Press, 1966.

[17] Thompson E., "Use of the pseudo - concentrations to follow creeping viscous flows during transient analysis", *Int.J.Num.Meth.Fluids*, **6**, pp.749-761, 1986.

[18] Thompson E. and Smelser R.E., "Transient analysis of forging operations by the pseudo - concentrations method", *Int.J.Num.Meth.Engrg.,* **25**, pp.177-189, 1988.

[19] Zienkiewicz O.C., Jain P.C. and Oñate E., "Flow of solids during forming and extrusion: some aspects of numerical solutions", *Int.J.Solid Struct.*, **14**, pp.15-28, 1977.

[20] Zienkiewicz O.C. and Taylor R.L., *The Finite Element Method* (4th. Edition), McGraw-Hill, 1989.